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Note

Spatial implementation ☆

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ABSTRACT

In a spatial model with Euclidean preferences, we introduce a new rule, the *geometric median*, and characterize it as the smallest rule (with respect to set inclusion) satisfying a collection of axioms. The geometric median is independent of the choice of coordinates and is Nash implementable.

1. Introduction

Our purpose is to study the implementation problem in spatial environments. We have in mind a spatial model where agents have Euclidean preferences. However, the ideal points or "peaks" of these preferences are unknown to the social planner. We study mechanisms that properly incentivize agents to reveal their peaks to the social planner, and which also satisfy other basic properties. The basic application we have in mind is the location of a public facility.

Ideally, the mechanism should achieve some very basic goals: some sort of social compromise should be reached and the participating agents should always have an incentive to reveal their true preferences over possible outcomes. The well-known Gibbard–Satterthwaite Theorem does not hold in this setting (Gibbard, 1973; Satterthwaite, 1975). Many papers study the strategy-proofness property in spatial environments with preferences that are "single-peaked."[1] Moulin (1980) provides the seminal result in this literature, focusing on the single-dimensional case. He establishes that the only rules which are efficient, anonymous, and strategy-proof select a "generalized" median of the announced peaks of the agents. Multi-dimensional extensions establish characterizations of coordinate-wise medians which hold even in the Euclidean case (see Border and Jordan, 1983; Kim and Roush, 1984; Barberà et al., 1993, 1998; Le Breton and Sen, 1999; Le Breton and Weymark, 1999; Bordes et al., 2012). Coordinate-wise medians are intuitive, but suffer from the drawback that the choice of coordinates determines the rule. In spatial environments involving land or distances, however, there is often no natural choice of coordinate. The notion of a median in all directions, or *total median*, first studied in the economics literature by Plott (1967), generalizes the ideas of Hotelling (1929) and Black (1948) to multiple dimensions. The median in all directions is independent of choice of coordinates. However, it seldom exists.

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[1] Massó and Moreno de Barreda (2011) provide a related characterization when preferences are also required to be symmetric.

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Strategy-proofness is in general incompatible with the notion that the outcome of a rule does not depend on choice of coordinates together with any reasonable anonymity properties. Hence, we ask for the weaker notion of Maskin Monotonicity (Maskin, 1999). Maskin monotonicity is known to be necessary for Nash implementability. Using this property, we will characterize a rule which turns out to be Nash implementable.

Our goal in this work is to investigate rules which do not necessarily depend on choice of coordinates, have reasonable incentive properties, and exist quite generally. We investigate several axioms, and establish that a new rule emerges, which naturally generalizes the classical median in one dimension. This rule is called the *geometric median*.[2] To understand its properties, consider the one-dimensional case. With an odd number of agents, the median is known to minimize the total absolute deviation from the ideal points: in our setting, it is a utilitarian optimum when agents preferences are represented by the Euclidean distance. A natural idea is to generalize this optimization problem to multiple dimensions. The result of this optimization in a multi-dimensional framework constitutes the geometric median. In general, there may not exist a unique point minimizing the total absolute deviation from the ideal points. Hence, the geometric median is a set-valued concept.

We investigate the geometric median in a variable population framework. In the case of an even number of agents, we know that there is not necessarily a unique geometric median. So, in this case, we characterize a set-valued concept. Our axiomatization is based on a concept we call *mean equality*, which we believe is new to the literature. It is based on the idea that if the mean is equidistant from every agent's ideal point, then it should be selected. This axiom incorporates the idea that the main reason the mean might not be selected is because it "unfairly" weights certain agents. The axiom suggests that when all agents are treated symmetrically by the mean, then the mean should be selected.

Our result also relies on two classical axioms. The first, *reinforcement*, is due to Young (1975).[3] Reinforcement requires that when two disjoint societies each select the same alternative, then the society which results from collecting the agents together should also select the same alternative. The other variable population axiom, which is related to reinforcement is *replication invariance*, which makes a first appearance in Debreu and Scarf (1963). The axiom requires that in replicating a society, the set of solutions should not change. Note that reinforcement already implies in replicating an economy all solutions of the original society should also be chosen for the replica. For our purposes, the important part of replication invariance is the converse: if an alternative is chosen for the replica, it must also be chosen for the original economy.

Finally, we postulate a technical axiom, which is formally an upper hemicontinuity axiom; we refer to it simply as continuity. We show that any rule satisfying these axioms must contain the set of geometric medians as possible solutions; further, the set of geometric medians viewed as a rule satisfies all of the axioms. Hence, the set of geometric medians is the smallest rule (with respect to set inclusion) satisfying the axioms.

Importantly, because the geometric median satisfies no veto-power, we know it is Nash implementable when there are at least three agents; see Maskin (1999). It is obviously Nash implementable when there is one agent, and it is straightforward to verify that the properties of Moore and Repullo (1990) or Dutta and Sen (1991) are satisfied, so that it is Nash implementable when there are only two agents. Hence, no matter the cardinality, it is a Nash implementable rule.

The paper is organized as follows. Section 2 presents the model and our main result. Section 3 concludes.

2. The model

Here we provide an axiomatization for a variable population model and show that a social choice rule selecting the (set of) geometric median(s) is the *smallest* rule satisfying our axioms, in the sense that any other rule satisfying our axioms must contain the set of geometric medians for any economy.

Let $X = \mathbb{R}^d$ be the policy space. For any $x, y \in X$ let $\|x - y\|$ denote the Euclidean distance.

Preferences are assumed to be Euclidean. Hence, any preference $\succsim$ can be represented by an "ideal point", $z \in X$, with the property that for any $x, y \in X$, $x \succsim y$ if and only if $\|x - z\| \leq \|y - z\|$.

Let $\mathbb{N}$ index the potential agents and $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. An *economy* (N, Z) is a set of agents $N \in \mathcal{N}$ and their ideal points $Z \in X^N$. Since there is a one-to-one relationship between preference profiles and a set of points in X, we will use the notation $Z \in X^N$ to indicate a preference profile of the agents that is represented by the points $Z = (z_i)_{i \in N}$ where $z_i \in X$ for each i.

Let the set of all possible economies be $\mathcal{E}$. A social choice rule is a correspondence $\varphi : \mathcal{E} \rightrightarrows X$ such that $\varphi(N, Z) \subseteq X$ for every economy (N, Z). Importantly, as is standard in the implementation literature, we allow φ to be empty-valued.

For $i \in N$ and $z_i, x \in X$, let $UC_i(z_i, x) = \{y \in X \mid \|y - z_i\| \leq \|x - z_i\|\}$ be the upper contour set for the preference relation represented by the point z_i at the point x. This is simply the set of all outcomes agent i weakly prefers to x. We will say that the preference relation represented by a point z_i' is a monotonic transformation of the preference relation represented by z_i at a point x if $UC_i(z_i', x) \subseteq UC_i(z_i, x)$. Let $MT(z_i, x)$ be the set of all monotonic transformations of the preference relation

[2] Despite its intuitive appeal in the spatial setting, the geometric median has received little attention in the social choice literature. See Chung and Duggan (2014) for a more general concept in the spatial model of voting. Cervone et al. (2012) investigate the geometric median in a preference aggregation framework. Finally, Baranchuk and Dybvig (2009) provide an application to corporate board consensus.

[3] For classical works using this property, see also Smith (1973), which deals with a preference aggregation environment, and the papers of Young characterizing specific rules; Young (1974); Young and Levenglick (1978).