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Stochastic bequest games

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ABSTRACT

In this paper, we prove the existence of a stationary Markov perfect equilibrium for a stochastic version of the bequest game. A novel feature in our approach is the fact that the transition probability need not be non-atomic and therefore, the deterministic production function is not excluded from consideration. Moreover, in addition to the common expected utility we also deal with a utility that takes into account an attitude of the generation towards risk.

1. Introduction

Deterministic bequest games were first discussed by Phelps and Pollak (1968) in the context of some considerations in the theory of economic growth. In their model, it is assumed that each generation lives, saves and consumes over just one period. Moreover, each generation cares about consumption of its immediate descendant and leaves it a bequest. This leftover part constitutes the next generation's inheritance that is determined by some continuous production function. We refer the reader to Maskin and Tirole (2001) and Chapter 13 in Fudenberg and Tirole (1991) for a further discussion of the concepts used and their interpretations. The existence of Markov perfect equilibria in such games in the class of left continuous strategies of bounded variation was proved independently by Bernheim and Ray (1983) and Leininger (1986). It is quite surprising that the proofs of the existence of equilibria in relatively simple models studied in Bernheim and Ray (1983) and Leininger (1986) are rather involved. Restricting attention to Lipschitz continuous or differentiable strategies makes no simplicity and leads to more restrictive assumptions; see Kohlberg (1976).

In this paper, we study a stochastic model of bequest games where the following generation's endowment is described by a stochastic transition probability function, which is assumed to be weakly continuous. It must be mentioned that this paper is not the first one that deals with probabilistic production function. Intergenerational games with stochastic transitions were already examined, for instance in Amir (1996), Nowak (2006), Balbus et al. (2012), Jaśkiewicz and Nowak (2014), where the existence of a stationary Markov perfect equilibrium was proved. However, all the aforementioned papers are concerned with pretty specific transitions, which are convex combinations of finitely many measures on the state space (see Remark 2). More interesting transition probability functions, e.g., determined by difference equations and some random i.i.d. shocks, were considered in Harris and Laibson (2001) and Balbus et al. (2014). However, a weak side of the approaches

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in Harris and Laibson (2001) and Balbus et al. (2014) is that the transitions are assumed to be non-atomic. Thus, the above-mentioned papers do not cover the deterministic case.

In this paper, the transition probability need not be non-atomic and therefore, the deterministic case is also included in our analysis. This is a new point, when compared with the above-mentioned papers. Another positive feature of this work is the fact that we consider two manners according to which a current generation derives its utility from its own consumption and that of its descendant. The first case refers to a standard expected utility, whereas the second case takes into account a risk attitude of the generation and employs the entropic risk measure; see Föllmer and Schied (2004). A similar criterion was proposed in Jaśkiewicz and Nowak (2014), but for games with the specific (and non-atomic) transition probabilities.

In our work, the set of possible endowments is an interval in the real line. The function space for searching for equilibria is the set of consumption strategies for which the corresponding investment functions are lower semicontinuous and non-decreasing (see Remark 3). Our proof of the existence of stationary Markov perfect equilibria for the two aforementioned utilities makes use of Fatou's lemma and the Skorohod representation theorem for weakly convergent probability measures. Using these tools, we make it relatively short compared with known proofs in the deterministic case given by Bernheim and Ray (1983) and Leininger (1986).

2. The model

Let R be the set of all real numbers, $\underline{R} := R \cup \{-\infty\}$, and N denote the set of all positive integers. Put $S := [0, \bar{s}]$ for some fixed $\bar{s} > 0$ and $S_+ := (0, \bar{s}]$. Define

$$A(s) := [0, s] \quad \text{and} \quad D := \{(s, a) : s \in S,\ a \in A(s)\}.$$

Consider an infinite sequence of generations labelled by $t \in N$. There is one commodity, which may be consumed or invested. Every generation lives one period and derives utility from its own consumption and a measure of consumption of its immediate descendant. Generation t receives the endowment $s_t \in S$ and chooses consumption at a level $a_t \in A(s_t)$. The investment of $y_t := s_t - a_t$ determines the endowment of its successor according to a transition probability q from S to S, which depends on $y_t \in A(s_t)$. If $s_t = s$, then we shall often write s' for s_{t+1}. The preferences of the current generation are represented by two functions: $u : S \mapsto \underline{R}$ and $v : S \mapsto R$ in a way that will be specified later. At the moment we assume that u and v are Borel measurable and v is bounded.

Let Φ be the set of Borel measurable functions $\phi : S \mapsto S$ such that $\phi(s) \in A(s)$ for each $s \in S$. A *strategy* for generation t is a function $c_t \in \Phi$. If $c_t = c$ for all $t \in N$ and some $c \in \Phi$, then we say that the generations employ a *stationary strategy*. The transition probability induced by q and $c \in \Phi$ is $q(\cdot|i(s))$, where $i(s) := s - c(s)$ is the *investment* or *saving* in state $s \in S$. Assume that generation t consumes $a \in A(s_t)$ in state $s_t = s$ and the following generation is going to use a strategy c. Then, the *utility* of generation t can be defined as follows

$$\widetilde{P}(a, c)(s) := u(a) + \beta \int_S v(c(s'))q(ds'|s - a), \tag{1}$$

where $\beta \in (0, 1]$ is a discount factor. This is a standard *expected utility* case. The solution concept given below comes from Phelps and Pollak (1968).

Definition 1. A *Stationary Markov Perfect Equilibrium* (*SMPE*) in the model with the expected utility is a function $c^* \in \Phi$, such that for every $s \in S$ we have

$$\sup_{a \in A(s)} \widetilde{P}(a, c^*)(s) = \widetilde{P}(c^*(s), c^*)(s).$$

Note that $(c^*, c^*, \ldots)$ is a Nash equilibrium in the game played by countably many generations.

We shall also study another model with a utility that reflects a generation's attitude towards risk. Assuming that generation t consumes $a \in A(s_t)$, $s_t = s$, and the following generation is going to use a strategy $c \in \Phi$, we can define the *utility* of generation t as follows

$$\widetilde{R}(a, c)(s) := u(a) + \frac{\beta}{r} \ln \int_S e^{rv(c(s'))} q(ds'|s - a), \tag{2}$$

where $\beta \in (0, 1]$ is a discount factor and $r < 0$ is a *risk coefficient.* This is a *utility involving the entropic risk measure,* see Example 4.34 in Föllmer and Schied (2004).

Definition 2. A *Stationary Markov Perfect Equilibrium* (*SMPE*) in the model with the utility involving the entropic risk measure is a function $c^* \in \Phi$, such that for every $s \in S$ we have

$$\sup_{a \in A(s)} \widetilde{R}(a, c^*)(s) = \widetilde{R}(c^*(s), c^*)(s).$$