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An extension of quasi-hyperbolic discounting to continuous time

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ABSTRACT

Two-Stage Exponential (TSE) discounting, the model developed here, generalises exponential discounting in a parsimonious way. It can be seen as an extension of Quasi-Hyperbolic discounting to continuous time. A TSE discounter has a constant rate of time preference before and after some threshold time; the switch point. If the switch point is expressed in calendar time, TSE discounting captures time consistent behaviour. If it is expressed in waiting time, TSE discounting captures time invariant behaviour. We provide preference foundations for all cases, showing how the switch point is derived endogenously from behaviour. We apply each case to Rubinstein's infinite-horizon, alternating-offers bargaining model.

A decision maker's rate of time preference is reflected in a variety of behaviours: smoothing consumption through saving, consuming or abstaining from tobacco, drugs or unhealthy food, investing in education, and so on. A constant rate of time preference, the predominant model of intertemporal choice in economics, rules out sudden changes in behaviour. Yet, people commonly resolve to start saving, quit smoking, eat better, and start exercising at some predetermined date. If utility is unchanged, it seems that their discount rate abruptly changes. This paper studies a model capturing this sudden change.

The *Quasi-Hyperbolic* (QH) discounting model elegantly captures a changing discount rate (Phelps and Pollak, 1968; Laibson, 1997; Hayashi, 2003; Attema et al., 2010; Montiel Olea and Strzalecki, 2014). Developed in discrete time, QH discounting involves weighting utility for outcomes using discount factors $\{1, \gamma\beta, \gamma\beta^2, \ldots\}$. QH discounting has been applied extensively in economic theory (Asheim, 1997; Laibson, 1997; Barro, 1999; O'Donoghue and Rabin, 2001; Luttmer and Mariotti, 2003). Extending QH discounting to continuous time is important for economic applications. One approach has been considered by Harris and Laibson (2013). This paper studies an extension of QH discounting called *Two-Stage Exponential* (TSE) discounting. TSE discounting provides a more robust, in a way we will describe, approach to modelling present-biased preferences.

As with exponential, and many other nonexponential discounting models (see Abdellaoui et al., 2010: 849), TSE discounting retains a stationary instant utility for outcomes. This utility is discounted by a constant rate of time preference up to a *switch point*. After this point, the discount rate may change, but then remains constant. Violations of constant discounting occur only when comparing the near and distant future. This parametric form of discounting was first presented by Jamison and Jamison (2011). We provide a preference foundation for TSE discounting over timed outcomes.

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We extend TSE discounting to dynamic choice by developing *time consistent* and *time invariant* (Halevy, 2012) versions of the model. It turns out that whether the model captures time consistent or time invariant behaviour depends only on the interpretation of one parameter, the switch point. If the switch point is expressed in *calendar time*, then the model is time consistent. If it is expressed in *waiting time*, then the model is time invariant. We apply each dynamic version of TSE discounting to the infinite-horizon, alternating-offers bargaining model of Rubinstein (1982). There are few previous studies of non-exponential discounting preferences in sequential bargaining (Akin, 2007; Ok and Masatlioglu, 2007; Noor, 2011; Kodritsch, 2012), all of which have assumed time invariance.

The outline of this paper is as follows: Section 1 contains the notation and definitions. Section 2 presents the exponential discounting model, 2.1 as applied to static choice and 2.2 as applied to dynamic decision making. In Section 3.1 we present the TSE discounting model and in Section 3.2 we give a preference foundation for the model. Section 4 then considers extensions of the TSE discounting model to dynamic choice. The time consistent version of the model is presented in Section 4.1 and preference foundations are given. The time invariant version of the model is presented in Section 4.2 and, again, preference foundations are given. Then, the models are applied to infinite-horizon, alternating-offers bargaining in Sections 5.1 and 5.2. All proofs are in the Appendices.

1. Definitions

Let $[0, X]$, with $X > 0$, denote the set of *outcomes* and $[0, T]$, with $T > 0$, be the set of times at which an outcome can occur. The set of *timed outcomes* is $[0, X] \times [0, T]$. A typical element of $[0, X] \times [0, T]$ is (x, t), which denotes the outcome x being received at time t. Such timed outcomes are the objects of choice.

A *static preference* relation $\succcurlyeq_{\underline{t}}$ is a binary relation defined over $[0, X] \times [\underline{t}, T]$; the set of timed outcomes occurring no sooner than time $\underline{t}$. A static preference relation characterises the preferences of our decision maker at time $\underline{t}$, as if they were making decisions at that time.[2] An *initial preference* relation $\succcurlyeq_0$ is a static preference relation for $\underline{t} = 0$. For a set of *decision times* $\mathscr{D} \subseteq [0, T]$ with $0 \in \mathscr{D}$, a *dynamic preference structure* $\mathscr{R} := \{\succcurlyeq_{\underline{t}}\}_{\underline{t} \in \mathscr{D}}$ is a set of static preference relations indexed by $\mathscr{D}$.

Given a static preference $\succcurlyeq_{\underline{t}}$, the relations $\succ_{\underline{t}}$, $\preccurlyeq_{\underline{t}}$, $\prec_{\underline{t}}$ and $\sim_{\underline{t}}$ are defined in the usual way. A static preference $\succcurlyeq_{\underline{t}}$ is *complete* if, for all $(x, t), (x', t') \in [0, X] \times [\underline{t}, T]$, at least one of $(x, t) \succcurlyeq_{\underline{t}} (x', t')$ or $(x', t') \succcurlyeq_{\underline{t}} (x, t)$ holds. It is *transitive* if, for all $(x, t), (x', t'), (x'', t'') \in [0, X] \times [\underline{t}, T]$, $(x, t) \succcurlyeq_{\underline{t}} (x', t')$ and $(x', t') \succcurlyeq_{\underline{t}} (x'', t'')$ jointly imply $(x, t) \succcurlyeq_{\underline{t}} (x'', t'')$. It is a *weak order* if it is complete and transitive. It is *monotonic* if, for all $(x, t), (x', t) \in [0, X] \times [\underline{t}, T]$, $(x, t) \succcurlyeq_{\underline{t}} (x', t)$ iff $x \geqslant x'$. It is *impatient* if, for all $(x, t), (x, t') \in [0, X] \times [\underline{t}, T]$, $(x, t) \succcurlyeq_{\underline{t}} (x, t')$ iff $t' \geqslant t$. We will always assume that $(0, t) \sim_{\underline{t}} (0, t')$, for all $t, t' \in [0, T]$, and include this condition in the definition of impatience. A static preference relation $\succcurlyeq_{\underline{t}}$ is *continuous* if, for all $(x, t) \in [0, X] \times [\underline{t}, T]$, the sets $\{(x', t') : (x, t) \succcurlyeq_{\underline{t}} (x', t')\}$ and $\{(x', t') : (x, t) \preccurlyeq_{\underline{t}} (x', t')\}$ are closed subsets of $[0, X] \times [\underline{t}, T]$.

A static preference relation $\succcurlyeq_{\underline{t}}$ is *represented* by a real-valued function $V_{\underline{t}} : [0, X] \times [\underline{t}, T] \to \mathbb{R}$ if, for all $(x, t), (x', t') \in [0, X] \times [\underline{t}, T]$, the following holds:

$$(x, t) \succcurlyeq_{\underline{t}} \left(x', t'\right) \quad \Leftrightarrow \quad V_{\underline{t}}(x, t) \geqslant V_{\underline{t}}\left(x', t'\right).$$

A necessary condition for $\succcurlyeq_{\underline{t}}$ to admit such a representation is that $\succcurlyeq_{\underline{t}}$ is a weak order. By Debreu (1964), weak ordering and continuity of $\succcurlyeq_{\underline{t}}$ are necessary and sufficient for the existence of a *continuous* utility representation. Monotonicity and impatience ensure that such a representation is non-decreasing in x and non-increasing in t.

We call a set of functions $\mathscr{V} := \{V_{\underline{t}}\}_{\underline{t} \in \mathscr{D}}$, where $V_{\underline{t}} : [0, X] \times [\underline{t}, T] \to \mathbb{R}$ for each $\underline{t} \in \mathscr{D}$, a *dynamic model*. Finally, we say that a dynamic preference structure $\mathscr{R}$ is *represented* by a dynamic model $\mathscr{V}$ if, for all $\underline{t} \in \mathscr{D}$, the preference relation $\succcurlyeq_{\underline{t}} \in \mathscr{R}$ is represented by $V_{\underline{t}} \in \mathscr{V}$.

2. Exponential discounting

2.1. Static choice and exponential discounting

This section reviews the classical exponential discounting model, as applied to initial or static choice over timed outcomes. Initial preferences conform to exponential discounting if they can be represented as follows:

$$V_0(x, t) = \delta^t u(x)$$

for all $(x, t) \in [0, X] \times [0, T]$, with $\delta \in [0, 1]$ and $u : [0, X] \to \mathbb{R}$ a continuous, strictly increasing function with $u(0) = 0$. The uniqueness properties pertaining to this representation are discussed later. The key property of exponential discounting is *stationarity*:

Definition *(Stationarity).* A static preference relation $\succcurlyeq_{\underline{t}}$ satisfies *stationarity* if for all $(x, t), (y, t + \tau), (x, s), (y, s + \tau) \in [0, X] \times [\underline{t}, T]$ the following holds:

$$(x, t) \succcurlyeq_{\underline{t}} (y, t + \tau) \quad \Leftrightarrow \quad (x, s) \succcurlyeq_{\underline{t}} (y, s + \tau).$$

[2] We underline the decision time, $\underline{t}$, as it becomes useful in presenting what follows.