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Epistemic equivalence of extended belief hierarchies ☆

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ABSTRACT

In this paper, we introduce a notion of epistemic equivalence between hierarchies of conditional beliefs and hierarchies of lexicographic beliefs, thus extending the standard equivalence results of Halpern (2010) and Brandenburger et al. (2007) to an interactive setting, and we show that there is a Borel surjective function, mapping each conditional belief hierarchy to its epistemically equivalent lexicographic belief hierarchy. Then, using our equivalence result we construct a terminal type space model for lexicographic belief hierarchies. Finally, we show that whenever we restrict attention to full-support beliefs, epistemic equivalence between a lexicographic belief hierarchy and a conditional belief hierarchy implies that an arbitrary Borel event is commonly assumed under the lexicographic belief hierarchy if and only if it is commonly strongly believed under the conditional belief hierarchy. This is the first result in the literature directly linking common assumption in rationality (Brandenburger et al., 2008) with common strong belief in rationality (Battigalli and Siniscalchi, 2002).

1. Introduction

A belief hierarchy describes an agents' beliefs, beliefs about every other agent's beliefs, and so on. Belief hierarchies are an integral part of modern economic theory, often used for analyzing games with incomplete information (Harsanyi, 1967/1968), as well as for providing epistemic characterizations for several solution concepts, such as rationalizability (Brandenburger and Dekel, 1987; Tan and Werlang, 1988), Nash equilibrium (Aumann and Brandenburger, 1995) and correlated equilibrium (Aumann, 1987), just to mention a few.[2]

A well-known problem of standard belief hierarchies is that they fail to capture conditional beliefs given zero probability events, and therefore they are not sufficiently rich to characterize solution concepts, such as iterated admissibility in normal form games or rationalizability in extensive form games, where unlikely – yet possible – events play a significant role. This difficulty has been circumvented in the literature by extending the notion of beliefs in two different ways that were developed independently.

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2 For an overview of the epistemic game theory literature we refer to the textbook by Perea (2012) or the review article by Brandenburger (2007).

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- According to the first approach, beliefs are captured by a *lexicographic probability system (LPS)*, which consists of a sequence of Borel probability measures, else called theories (Blume et al., 1991a). The primary theory coincides with the standard beliefs, the secondary theory captures the beliefs once the agent has for some reason discarded the primary theory, and so on. Extending this construction to an interactive setting gives rise to a hierarchy of lexicographic beliefs ($\mathcal{L}$-hierarchy): The first order lexicographic beliefs consist of an LPS over the underlying space of uncertainty, the second order lexicographic beliefs consist of an LPS over the underlying space of uncertainty and the opponents' first order lexicographic beliefs, and so on. Hierarchies of lexicographic beliefs have been used to epistemically characterize several solution concepts in normal form games, such as iterated admissibility (Brandenburger et al., 2008), self-admissible sets (Brandenburger and Friedenberg, 2010), perfect equilibrium and proper equilibrium in two-players normal form games (Blume et al., 1991b) and proper rationalizability (Asheim, 2001; Perea, 2011).
- According to the second approach, beliefs are captured by a *conditional probability system (CPS)*, which consists of a collection of conditioning events and a conditional Borel probability measure given each conditioning event, in a way such that Bayes rule is satisfied whenever possible. In dynamic games, a conditioning event typically corresponds to an information set. Extending this idea to an interactive setting induces a hierarchy of conditional beliefs ($\mathcal{C}$-hierarchy): The first order conditional beliefs are described by a CPS over the underlying space of uncertainty, the second order conditional beliefs consist of a CPS over the underlying space of uncertainty and the opponents' first order conditional beliefs, and so on. Conditional belief hierarchies have been widely used to characterize solution concepts in dynamic games, such as extensive form rationalizability (Battigalli and Siniscalchi, 2002) and extensive form best response sets (Battigalli and Friedenberg, 2012).

Several authors have studied the relationship between the two models (Brandenburger et al., 2007; Halpern, 2010). As it turns out, the two approaches are epistemically equivalent, in the sense that there exists a surjective mapping from the space of CPS's onto the space of LPS's (Brandenburger et al., 2007).[3]

In this paper, we extend this idea to an interactive setting, by introducing a notion of epistemic equivalence between $\mathcal{L}$-hierarchies and $\mathcal{C}$-hierarchies, thus providing a stepping stone for understanding the relationship between solution concepts whose epistemic characterizations use different types of belief hierarchies. The importance of establishing a notion of epistemic equivalence has been already pointed out in a different context (Brandenburger and Friedenberg, 2010).

Our extension is far from trivial, as the previously defined notion of epistemic equivalence relates CPS's and LPS's that are defined on the same space. However, second order conditional beliefs are described by a CPS over the underlying space of uncertainty and the opponents' first order conditional beliefs, whereas second order lexicographic beliefs are described by an LPS over the underlying space of uncertainty and the opponents' first order lexicographic beliefs. Thus, in order to introduce a notion of epistemic equivalence between second order beliefs, we first need to translate each Borel event in the space of first order lexicographic beliefs to a Borel event in the space of first order conditional beliefs. In fact, we do this by showing that the surjective function that maps CPS's onto LPS's is Borel measurable (Lemma 2). Then, second order conditional beliefs are mapped surjectively onto second order lexicographic beliefs via a Borel function, which allows us in turn to define epistemic equivalence between third order beliefs. Continuing inductively, we show that there is a Borel surjective function, mapping each conditional belief hierarchy to a lexicographic belief hierarchy (Theorem 1).

Using our main equivalence result, together with the existence of a universal type space for $\mathcal{C}$-hierarchies (Battigalli and Siniscalchi, 1999, Prop. 2), we indirectly construct a terminal type space model for lexicographic belief hierarchies, i.e., an LPS-based type space model that induces all $\mathcal{L}$-hierarchies (Theorem 3). To the best of our knowledge, this is the first such result in the literature, and it provides a Bayesian foundation for hierarchies of lexicographic beliefs.

The natural analogue of probability-1 belief in a CPS is strong belief (Battigalli and Siniscalchi, 1999), while in an LPS the corresponding notion is assumption (Brandenburger et al., 2008). One natural question arising then is whether our concept of epistemic equivalence also implies equivalence between common strong belief and common assumption, i.e., if a Borel event is commonly strongly believed under a conditional belief hierarchy, is it also commonly assumed under the epistemically equivalent lexicographic belief hierarchy, and vice versa? Brandenburger et al. (2007) have already shown that if we restrict attention to full-support beliefs in a single-agent framework, a Borel event is strongly believed under a CPS if and only if it is assumed under the epistemically equivalent LPS. Generalizing this result to our interactive setting, we prove that if beliefs are full-support, a Borel event is indeed commonly strongly believed under a $\mathcal{C}$-hierarchy if and only if it is commonly assumed under the epistemically equivalent $\mathcal{L}$-hierarchy (Proposition 4). Notice that this result applies not only to events in the underlying space of uncertainty, but also to events that involve beliefs, such as rationality, implying that it has interesting implications for solution concepts in games, e.g., for clarifying the relationship between common assumption in rationality (Brandenburger et al., 2008) and common strong belief in rationality (Battigalli and Siniscalchi, 2002).

The paper is structured as follows: Section 2 contains some necessary mathematical preliminaries; Section 3 introduces the notions of conditional probability systems and lexicographic probability systems, and presents the existing notion of epistemic equivalence between the two; Section 4 defines conditional belief hierarchies and lexicographic belief hierarchies;

[3] For a precise definition of epistemic equivalence, see Definition 5.