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Note

Priority matchings revisited ☆

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ABSTRACT

We consider a pairwise kidney exchange model. Roth et al. (2005) define priority matchings of the model and introduce a mechanism to derive them. In this paper, we re-examine the priority matching. First, we consider a general priority ordering where multiple patients may hold equal priority. We provide a characterization of the priority matchings by using the concept of alternating paths. Using the characterization, we examine the effect of a small change in the priority order on a set of priority matchings. Moreover, we provide an efficient method to find a priority matching.

1. Introduction

This paper considers a pairwise kidney exchange model that is firstly discussed by Roth et al. (2005).[2] In the model, there are kidney patients each of whom has one incompatible donor. Two patient-donor pairs can exchange if each donor is compatible for the patient paired with the other donor. Roth et al. (2005) discuss how to organize such exchanges. They introduce a priority mechanism to obtain a priority matching that matches as many patients as possible starting with the patient with the highest priority and following the priority ordering, never sacrificing a higher priority patient for a lower priority patient. One of the reasons why they consider the mechanism is that in many situations, kidney patients may be ordered by a natural priority ordering. For example, they state that the sensitivity of a recipient to the tissue types of others, known as panel reactive antibody (PRA), can be one of the criteria to determine the priority ordering.[3]

In this paper, we reconsider the priority matchings. Roth et al. (2005) assume that no two patients have the same priority order. On the other hand, we consider a more general priority ordering; that is, where multiple patients can follow the same priority order. Moreover, we characterize priority matchings by using the concept of alternating paths, which has attracted adequate attention in the literature on graph theory. Using this characterization, we consider the effect of a small change

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[2] This model is equivalent to a specific roommates model where every agent evaluates all other agents as acceptable or unacceptable. Chung (2000) and Bogomolnaia and Moulin (2004) call such a specific preference of an agent a dichotomous preference. That is, our model is equivalent to a roommates model with dichotomous preferences.

[3] There is a growing body of literature on the kidney exchange problem. See Sönmez and Ünver (2011) for a survey of this literature.

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in the priority order on a set of priority matchings. For example, consider the case where PRA is the unique criterion to determine the priority order. Suppose that the PRA of a patient is merely higher than but almost the same as that of one of the other patients. Then, should we try to interchange their priorities? In this paper, we try to answer to the question.

Further, we provide an efficient method to find a priority matching. It is too difficult to find a priority matching by the method introduced by Roth et al. (2005) when the number of patients is large. To find a priority matching using their method, we need to list all possible matchings. However, no study has introduced any efficient method to list the matchings. This implies that we may not be able to derive any priority matching within reasonable time when the number of patients is large. However, by using our method, we can derive a priority matching within reasonable time even in the worst case. In other words, a polynomial-time algorithm to have a priority matching is provided in this paper.[4]

2. Model

Let $N = \{1, 2, \cdots, n\}$ be a finite set of patients each of whom has one or more incompatible donors. A patient has a dichotomous preference; that is, he/she evaluates each of the other patients as acceptable or unacceptable. That is, if at least one (all) donor(s) of patient j is (are) compatible (incompatible) for patient i, then i evaluates j as acceptable (unacceptable).

We consider the following preference relation $\succsim_i$ for each patient i over N. First, if a patient i evaluates the other patient j acceptable, then $j \succ_i i$, where $\succ_i$ denotes the strict preference relation of i. Second, if a patient i evaluates the other patient k unacceptable, then $i \succ_i k$. Third, if $j \succ_i i$ and $k \succ_i i$, then $j \sim_i k$, where $\sim_i$ denotes the indifference relation of i. Fourth, if $i \succ_i j$ and $i \succ_i k$, then $j \sim_i k$. That is, if a patient i evaluates two patients as acceptable (or unacceptable), then i is indifferent between the two patients.

If two patients consider each other as acceptable; that is, if $j \succ_i i$ and $i \succ_j j$, then the patients are said to be *mutually compatible*. Otherwise; that is, if $i \succ_i j$ or $j \succ_j i$ (or both), then the patients are said to be *mutually incompatible*.

It should be noted that this model can be thought as a roommate model with dichotomous preference. In that case, N is the set of students that can be matched in pairs to be roommates in a student accommodation. Each student evaluates each of the other students as acceptable or unacceptable. Note also that the model includes a two-sided matching under dichotomous preferences as a special case. For example, when we consider a marriage model, if i and j are persons of the same sex, then they are mutually incompatible. Bogomolnaia and Moulin (2004) discuss the two-sided matching model.

We represent the mutual compatibility and incompatibility by using a simple undirected graph. Let ij be an edge between two patients $i, j \in N$. We denote that i and j are the *endpoints* of ij. Let g be a set of edges where $ij \in g$ if and only if i and j are mutually compatible. In this paper, we assume that each patient is an endpoint of at least one edge in g.

We call a subset of g a *subgraph*. If a patient is an endpoint of an edge in a subgraph, then we denote that the patient *belongs to* the subgraph. A *matching* M is defined as a subgraph of g such that for all $i, j \in N$, if $ij \in M$, then $ik \notin M$ and $jk \notin M$ for all $k \in N \setminus \{i, j\}$. Since a matching M is a subgraph of g, any i and j such that $ij \in M$ are mutually compatible and therefore each matching is individually rational; that is, $j \succ_i i$ and $i \succ_j j$. We denote that $i \in N$ is *matched* (*unmatched*) in M if $ij \in (\notin) M$ for one (all) $j \in N$. Moreover, the sets of matched patients and unmatched patients in M are given by $N_1(M)$ and $N_0(M)$, respectively. Let $|A|$ be the number of the elements of a set A. Then, $|N_1(M)| = 2|M|$ and $|N_0(M)| = n - 2|M|$ are satisfied. That is, the number of matched patients is double of the number of matches. Let $\mathcal{M}$ be the set of all possible matchings. Note that $|M| \leq n/2$ for all $M \in \mathcal{M}$; that is, the number of matches are less than or equal to the half of the number of patients.

Let $M \triangle M'$ be the *symmetric difference* between M and M' which is the set of elements either in M or M' but not in $M \cap M'$. For example, if $M = \{12, 34\}$ and $M' = \{34, 56\}$, then $M \triangle M' = \{12, 56\}$.

A subgraph $\{i_1 i_2, i_2 i_3, \cdots, i_{K-1} i_K\} \subseteq g$ is called a *path* if $i_1, i_2, \cdots, i_K$ are distinct, where $K - 1 \in [1, n-1]$ represents the length of the path. The following paths are keys to our results.

Definition 1. Fix $M \in \mathcal{M}$. A path $\{i_1 i_2, i_2 i_3, \cdots, i_{K-1} i_K\} \subseteq g$ is an M-alternating path if $i_{2a} i_{2a+1} \in M$ for all $a = 1, \cdots, \bar{a}$, where $\bar{a} = (K-1)/2$ if K is odd and $\bar{a} = (K-2)/2$ if K is even. Moreover, if the length of an M-alternating path is odd (or equivalently K is even), then it is called an M-augmenting path.

For example, let $g = \{12, 16, 23, 34, 45, 67, 78\}$ and $M = \{23, 45, 67\}$. Then, $\{12, 23, 34, 45\}$ and $\{16, 67, 78\}$ are M-alternating paths and the latter is an M-augmenting path.

A matching $M \in \mathcal{M}$ is said to be *Pareto dominated* by $M' \in \mathcal{M}$ if there is at least one patient i such that $i \in N_0(M)$ and $i \in N_1(M')$, and for all $j \in N_1(M)$, $j \in N_1(M')$. That is, if $N_1(M) \subsetneq N_1(M')$, then $M \in \mathcal{M}$ is Pareto dominated by $M' \in \mathcal{M}$.

[4] For more on the importance of polynomial-time algorithms, see Korte and Vygen (2005, Ch. 1).