Games and Economic Behavior 81 (2013) 11–26

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

The price of imperfect competition for a spanning network [☆]

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A R T I C L E   I N F O

Article history:
Received 31 March 2011
Available online 24 April 2013

JEL classification:
C72
C79
D40
D43
D44

Keywords:
Algorithmic mechanism design
Worst case scenario equilibrium analysis
Frugality
Minimum cost spanning tree problem
Price of imperfect competition

A B S T R A C T

A buyer procures a network to span a given set of nodes; each seller bids to supply certain edges, then the buyer purchases a minimal cost spanning tree. An efficient tree is constructed in any equilibrium of the Bertrand game.

We evaluate the *price of imperfect competition* (PIC), namely the ratio of the total price that could be charged to the buyer in some equilibrium, to the true minimal cost. If each seller can only bid for a single edge and costs satisfy the triangle inequality, we show that the PIC is at most 2 for an odd number of nodes, and at most $2\frac{n-1}{n-2}$ for an even number n of nodes. Surprisingly, this worst case ratio does not improve when the cost pattern is ultrametric (a much more demanding substitutability requirement), although the overhead is much lower on average under typical probabilistic assumptions. But the PIC increases swiftly when sellers can only provide a subset of all edges.

1. Introduction

Bertrand competition for differentiated commodities typically implies some welfare losses, as in the Hotelling model. We consider a special case where it does not, and where the surplus that the competing firms are able to extract admits a simple upper bound.

A buyer procures a network spanning a given set of nodes, while the sellers bid for different edges of the network. Efficiency requires to build a minimal cost spanning tree. This familiar optimization problem has a variety of applications (including rail infrastructure, the Internet's backbone, water distribution, etc.; see Sharkey, 1995, for a survey).

If one seller is the sole bidder for a certain edge, or a group of edges, he can typically bid higher than their true cost. But this overhead is bounded above because the edges are partial substitutes: for any edge e, there are several alternative paths ensuring the connection of e's two end nodes. We wish to evaluate the welfare consequences of this *imperfect competition* between the sellers.

We observe first that irrespective of the ownership structure and of the cost pattern, in equilibrium an efficient (minimal cost) spanning tree is constructed (Proposition 1). This is very close to being a special case of the efficiency result in the matroid markets of Chen and Karlin (2007) – on which more below.

☆ Several comments by two anonymous referees, an associate editor, Tim Roughgarden, Jay Sethuraman, Vijay Vazirani, and seminar participants at CORE, CoED 2011, Louisiana State, Midwest Theory 2011, Montreal, Rochester, Texas A&M, and U. de los Andes have been very helpful. Moulin's research is supported by the NSF under Grant CCF 1101202. Velez thanks the Melbern G. Glasscock Center for Humanities Research at Texas A&M University for financial support. All errors are our own.

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0899-8256/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.geb.2013.03.012

Our main concern is to evaluate the surcharge to the buyer procuring the network. We provide tight bounds when the cost of the edges satisfy the triangle inequality, i.e., for any three edges e, f, g forming a triangle the cost of one edge is not larger than the sum of the costs of the other two. We speak then of *metric* costs. This familiar assumption is realistic when costs are a subadditive function of the Euclidean length of an edge, or some other measure of spatial distance between its end nodes (for instance the shortest length of a path connecting the nodes in an arbitrary network).[1] It is also automatic when all sellers have access to the same technology to build edges, but each one is only licensed to bid for certain edges: it is then feasible to connect the end nodes of e by building f and g.

Our first main result (Theorem 1) is that if each seller bids for a single edge, there is at least one bidder for each edge, and true costs are metric, the buyer's surcharge essentially cannot exceed 100% of the true minimal cost. However our second main result (Theorem 2) says that the overhead grows rapidly if some edges are not available (no one is bidding for those edges): if at least half of the edges are missing, the worst case ratio of the overhead to the true cost is $n-1$, where n is the number of nodes to connect.

It is instructive to compare our results to those of the literature evaluating the *frugality* of some related auction mechanisms (Archer and Tardos, 2001, 2002; Karlin et al., 2005; Talwar, 2003). The procurement of a spanning tree among sellers who each own an edge of the network is a special case of the procurement of a matroid basis; other examples include the procurement of a path between two given nodes of the network (Archer and Tardos, 2002; Elkind et al., 2004; Immorlica et al., 2005), and more generally the procurement of a team to perform a complex task (Karlin et al., 2005). Like us, these papers are poised to evaluate the worst possible surcharge to the buyer. However, the payments to the various sellers are more complicated than in the simple first-price auction of Bertrand competition: they are given by the canonical VCG mechanism (known as the *pivotal* mechanism since Green and Laffont, 1979), in which each winning edge is paid its true cost, plus the extra cost incurred if we cannot use that very edge in our spanning tree. Under this payment scheme bidding one's true cost is a dominant strategy for each seller. Thus, implementation of the mechanism is *prior-free*, in sharp contrast with our equilibrium analysis of the Bertrand game requiring complete information among sellers.[2]

When we assume, as we do in our main results, that a seller bids for a single edge, it turns out that the two games, Bertrand bidding and pivotal, are essentially equivalent: the buyer's surcharge in the pivotal game is precisely the same as in the most expensive equilibrium (equivalently, in its unique equilibrium in undominated strategies) of the Bertrand game (see Section 6.1). Thus, our results can be used indifferently in both contexts.

There have been at least three attempts in the literature to measure the frugality of a mechanism to procure a spanning tree. They differ in the choice for the benchmark cost to which the total payment of the seller is compared. In Archer and Tardos (2001, 2002), like here, the benchmark is the true minimal (efficient) cost. They show that the worst case ratio (total charge to efficient cost) is again $n-1$, where n is the number of nodes to connect.[3] Our Theorem 1 qualifies this negative result when costs are metric.

Subsequently, Talwar (2003) uses for benchmark the cheapest cost of a spanning tree with no edge in common with the efficient tree. This curious proposal may lead in our model to a frugality ratio smaller than one[4]! Finally in Karlin et al. (2005) the benchmark is the solution of a linear program that, in the spanning tree problem, coincides with the most expensive equilibrium (for the buyer), so that the frugality ratio is one, tautologically.

We submit that from the buyer's perspective, the only meaningful benchmark is the true efficient cost. To avoid confusion with the multiple frugality indices, and to convey the basic economic intuition, we propose to call *price of imperfect competition* (PIC) the ratio of the worst buyer's charge in *some* equilibrium (or the actual charge in the pivotal mechanism), to the efficient cost.[5]

Summary of results. Section 2 introduces the minimum cost spanning tree problem and the Bertrand game of procurement. Proposition 1 states that an efficient tree is built in all relevant equilibria. Starting with Section 3 we assume that each seller bids for only one edge. We provide a general formula for the PIC that uses only the network structure and arbitrary costs (Proposition 2 and Corollary 1).

We assume in Section 4 that costs are metric, and show (Theorem 1) that if all edges have at least one bidder the worst case PIC is 2 for an odd number of nodes, and $2\frac{n-1}{n-2}$ for an even number n. However a single missing edge raises the worst case PIC to 3 (Proposition 5), and more generally the PIC may reach $n-1$ as soon as half of the edges are missing (Theorem 2).

In Section 5 we assume that costs are *ultrametric*, i.e., if edges e, f, g form a triangle, $c_e \leqslant \max\{c_f, c_g\}$. This is stronger, and implies much closer substitutability than the triangle inequality. Ultrametric costs are a natural restriction in some environments. Consider a set of nodes for which the connection cost is determined by the compatibility of certain attributes

[1] The cost of e takes the form $f(\|e\|)$, where f has increasing returns and $f(0)=0$.

[2] As well as the technical twist of the limit equilibrium, see Section 2.3.

[3] Assume costs are infinite outside a single cycle connecting all n nodes, and on the cycle all costs are zero except for a single edge with cost 1. Then the ratio is $n-1$.

[4] Assume 4 nodes a, b, d, e with metric costs $c_{ab}=c_{bd}=c_{de}=1$, $c_{ad}=c_{be}=2$, $c_{ae}=3$. Efficient cost is 3, the total pivotal charge is 6, and the cost of the only edge disjoint tree is 7.

[5] This terminology is obviously inspired by the notions *price of anarchy* and *price of stability* (Koutsoupias and Papadimitriou, 1999; Nisan et al., 2007). But the PIC measures the impact of decentralizing the procurement process on the buyer's welfare only. In our simple model imperfect competition induces no aggregate welfare loss.