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Best-reply dynamics in large binary-choice anonymous games

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ABSTRACT

We consider small-influence anonymous games with a large number of players n where every player has two actions. For this class of games we present a best-reply dynamic with the following two properties. First, the dynamic reaches Nash approximate equilibria fast (in at most $cn\log n$ steps for some constant $c>0$). Second, Nash approximate equilibria are played by the dynamic with a limit frequency of at least $1-e^{-c'n}$ for some constant $c'>0$.

1. Introduction

In this paper we consider an important class of *anonymous games* introduced by Schmeidler (1973) in the settings of stochastic dynamics. In this class of games the payoff of each player depends only on his own action and on the aggregate behavior of the other players, rather than the exact action profile of the other players. Anonymous games have received much attention in various fields. In economics, for example, the market price depends on the total amount of production. In computer science, congestion games are anonymous games because the cost of an edge depends on the total number of users there. In biology, single-population games are anonymous games.

Assume that a large number of players n should decide whether to use technology A or technology B. The payoff function of every player depends on the aggregate behavior of A and B users, and is a λ-Lipschitz function of the aggregate proportion of A users.[1] Note that unlike population games, here the payoff functions of two different players could be different. At every step a single player is chosen uniformly at random;[2] then the chosen player updates his action according to his best reply to the *observable aggregation*. We assume that the observable aggregation is not the exact aggregation but a rounding function of it (e.g., the integer percentage of players using technology A); this assumption could be explained, for example, by the difficulties (or costs) of computing the exact aggregation. Our main results claim that in $O(n\log n)$ steps in expectation the played action profile will be a pure Nash approximate equilibrium (Theorem 1); in addition, most of the time the played action profiles will be pure Nash approximate equilibria (Theorem 2).

The analysis of the convergence and its rate is composed of two pieces. The first piece (Proposition 1) is to show that the aggregate behavior converges to a *stable value* (see Definition 1). This stable value enjoys the property that there exists a pure Nash approximate equilibrium where the aggregate behavior is the stable value. Now, the second piece (Proposition 3) is to show that the dynamic reaches such an equilibrium before the aggregate behavior moves far from the stable value.

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[1] Since there are only two actions, the aggregate behavior could be represented by a single parameter of A users.

[2] For example, if the players play in continuous time and the updating opportunities for players are governed by i.i.d. Poisson arrival processes then it leads exactly to the case where the chosen player will be picked uniformly.

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The first piece of analysis is based on random-walk arguments. In order to derive the desired bounds on the transition probabilities of the aggregate behavior we use Andre's reflection principle. The second piece of analysis is based on the following observation: if all players updated their action according to their best reply while the aggregation did not change significantly, then the dynamic reaches a pure Nash approximate equilibrium. The analysis uses coupon collector arguments in order to derive $n \log n$ rate of convergence.

The paper is organized as follows. Section 2 discusses related literature. Section 3 defines formally the setup and the results. In Section 4 we informally introduce the ideas of the proof. The formal proof appears in Section 5. Section 6 is devoted to discussion and remarks.

2. Related literature

Anonymous games have been the object of study for over three decades. Schmeidler (1973) first considered non-atomic games with a continuum of players. Rashid (1983) studied the discretization of Schmeidler's result for a finite number of players. Azrieli and Shmaya (2011) proved that every *small-influence* game where a deviation of player i has a small influence on the payoff of player j possesses a PNE_ε; in particular, λ-Lipschitz anonymous games belong to this class.

A function that appears in those papers (and actually it appears in almost every paper that concerns anonymous games) is the *aggregated best-reply correspondence* that maps every aggregate behavior p into the aggregate behavior where all players best-replying to p.

Binary-choice anonymous games have been considered in a deterministic dynamics framework by Blonski (1999). He considered a continuum of players facing a binary-choice anonymous games in continuous time. Blonski (1999) considers only *heterogeneous* games; i.e., games where for every aggregate behavior the set of indifferent players has measure 0. The heterogeneity assumption yields a single valued and continues aggregated best-reply correspondence. Under the heterogeneity assumption Blonski introduces a deterministic dynamic that converges to a pure Nash equilibrium.

Ely and Sandholm (2005) considered general anonymous games in Bayesian settings where there is a continuum of players of each type. The assumptions in Ely and Sandholm (2005) were similar to those in Blonski (1999), they assumed that the aggregated best-reply correspondence is single valued and Lipschitz. Similar to Blonski (1999) Ely and Sandholm show that the deterministic best-reply dynamic converges to a Bayesian pure Nash equilibrium.

This paper is a discrete version of Blonski (1999) and Ely and Sandholm (2005), the set of players is finite, and we consider discrete-time dynamic that is *stochastic*. The analysis in the discrete stochastic case becomes much more complicated because of the following reasons. First, since our dynamic is stochastic, it may move to the "wrong direction" with positive probability, whereas the dynamics in Blonski (1999) and Ely and Sandholm (2005) deterministically moves into the "correct direction". Second, unlike Blonski (1999) and Ely and Sandholm (2005) we do not assume any continuity of the aggregated best-reply correspondence.[3] We overcome the discontinuity problem by assuming that every player observes the rounded aggregation. This makes the aggregated best-reply correspondence to be piece-wise constant.

Daskalakis and Papadimitriou (2007) prove that there exists an efficient algorithm for computing pure Nash approximate equilibrium (PNE_ε) in anonymous games using the fact that PNE_ε are obtained from fixed points of the aggregated best-reply correspondence. Application of the algorithm of Daskalakis and Papadimitriou to our case of two-action games generates a communication protocol that requires $1/\varepsilon$ steps of communication in order to reach PNE_ε. Although the communication procedure reaches PNE_ε very fast, it is nevertheless far from reflecting a natural adaptive behavior. In other words, the algorithm of Daskalakis and Papadimitriou is computationally very efficient, but it does not solve the problem from a dynamical perspective since it is an "unnatural" exhaustive search.

Friedman et al. (2009) consider a simultaneous best-reply dynamic (i.e., the best-reply updating is carried out simultaneously for all players) in anonymous games. They consider environments where players may not know the aggregate behavior, and may not know their own payoff function. They prove that this uncertainty might be overcome by stage learning. Their result proves that in games where the simultaneous best-reply dynamic converges to *PNE*, so does the best reply that uses stage learning. In addition, the rate of convergence in both cases is the same.

Hart and Mansour (2010) considered dynamics with large number of players n. Their negative result shows that for every dynamic satisfying the natural requirement of uncoupledness (i.e., dynamics where the action of every player i at every step depends on the payoff function of player i and the past realized action profiles, but does *not* depend on the payoff functions of players $j \neq i$) there exists a game where it will take $\exp(n)$ periods in average to reach a Nash equilibrium. This impossibility result leads us to the conclusion that in order to find a dynamic that converges fast to Nash equilibria in games with a large number of players we must consider some subclass of games. Classes of large games where convergence to Nash equilibria is known to be fast are potential games (Awerbuch et al., 2008), population acyclic games in evolutionary game theory (Arieli and Young, 2011), and some specific games, see for example Fischer et al. (2010) and Fischer and Vöcking (2004). This paper shows that also for large binary-choice anonymous games convergence to equilibrium could occur fast.

[3] A natural tool for proving convergence in stochastic discrete settings using continuous deterministic settings is the stochastic approximation method (see, e.g., Benaim and Weibull, 2003). The fact that we do not assume any continuity makes it impossible to imply stochastic approximation method on the results in Blonski (1999) or Ely and Sandholm (2005).