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# Absorbing sets in roommate problems [☆]

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ABSTRACT

We analyze absorbing sets as a solution for roommate problems with strict preferences. This solution provides the set of stable matchings when it is non-empty and some matchings with interesting properties otherwise. In particular, all matchings in an absorbing set have the greatest number of agents with no incentive to change partners. These "satisfied" agents are paired in the same stable manner. In the case of multiple absorbing sets we find that any two such sets differ only in how satisfied agents are matched with each other.



## 1. Introduction

Matching markets are of great interest in a variety of social and economic environments, ranging from finding a marital partner through pairing students to share rooms at colleges to matching workers with firms.[1] One of the goals of analyzing these markets is to find those matchings that are stable. Loosely speaking, a matching is stable if there are no two agents who prefer being with each other to being matched with their current partners. A frequent problem in most matching markets is that there may be no stable matchings. In such cases it is quite common to model matching markets by restricting the preference domain through assumptions that guarantee stability.[2] We believe, however, that applying solution concepts to the entire domain of the problem under study could help further our understanding of the performance of matching markets.

We focus our attention on the one-sided matching problem known as the roommate problem, in which each agent with preferences over all agents is allowed to form at most one partnership. This is the canonical problem concerned with lack of stability in the literature on matching. Additionally, it generalizes the marriage problem, whose set of agents is divided into two disjoint subsets and in which each agent accepts matching only with an agent from the other subset. Both marriage and roommate problems were introduced by Gale and Shapley (1962), who show that the latter may (unlike the former) have no stable matchings. Roommate problems that do not admit stable matchings are called unsolvable while all others are said to be solvable.

[☆] A preliminary version of the paper circulated under the title: "The stability of the roommate problem revisited".

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[1] For a comprehensive survey of matching markets see Roth and Sotomayor (1990).
[2] See for example Roth (1985) and Kelso and Crawford (1982).

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The aim of this paper is to study an alternative notion of stability in the entire class of roommate problems with strict preferences, with special emphasis on unsolvable problems. Our interest is enhanced from the empirical perspective in that, as Pittel and Irving (1994) observe, when the number of agents increases the probability of a roommate problem being solvable decreases fairly steeply.

The approach followed in this paper is to associate each roommate problem with an abstract system formed by the set of all matchings and a binary relation defined over this set. This binary relation corresponds to the standard blocking definition and can be described as follows: a matching "directly dominates" another matching if it can be obtained from the other by satisfying a blocking pair. We derive the absorbing sets of this system to analyze the stability of the roommate problem.[3] In this setting, an absorbing set can be defined as a set of matchings that satisfies the following two conditions: (i) any two distinct matchings within the set dominate[4] each other and (ii) no matching in the set is directly dominated by a matching outside the set. We believe that the selection of this solution concept is well-justified since, on the one hand, if the roommate problem is solvable it provides the matchings in the core, that is, it satisfies *core-consistency*. On the other hand, if the roommate problem is unsolvable it gives some sets of matchings that satisfy the property of *outer stability*, that is, matchings that are not in an absorbing set are dominated by the matchings of an absorbing set.

The approach followed and the notion of absorbing set may perhaps be better understood with the following description: Consider a dynamic process in which an unstable matching is adjusted when a blocking pair of agents mutually decide to become partners. Either this change gives a stable matching or a new blocking pair of agents will generate another matching and so on. If there are any stable matchings this dynamic process eventually converges to one of them, as shown by Diamantoudi et al. (2004). Otherwise the process will lead to a set of matchings (an absorbing set) such that via this dynamic process any matching in the set can be obtained from any other (Condition (i)) and it is impossible to escape from the set of matchings in the absorbing set (Condition (ii)). As Kalai and Schmeidler (1977) point out, an abstract system can be thought of as a Markov process where one alternative dominates another if and only if there is a positive probability of transition from the latter to the former. Following this line of research, Klaus et al. (2010) show that the set of stochastically stable matchings of a roommate problem with strict preferences coincides with the matchings in the union of absorbing sets.

This paper provides a characterization of absorbing sets in terms of stable partitions.[5] The importance of stable partitions lies in the fact that they identify specific structures in agents' preferences that provide the necessary and sufficient condition for the existence of stability. Using this notion we establish that stable partitions generate the matchings in absorbing sets (Proposition 1). We use this result to prove that in solvable problems each absorbing set consists of a unique stable matching (Proposition 2).

Unfortunately not all stable partitions are "generators" of absorbing sets, but we find a way to identify those that are. We observe that for each stable partition agents can be split into satisfied and dissatisfied, where satisfied agents are those with no incentive to change partners. We find that the stable partitions that induce the absorbing sets are those with the maximal set of satisfied agents (Theorem 1). We also show that this set is the same for all these stable partitions (Proposition 3). This characterization allows us to show some properties of the matchings within an absorbing set. In particular, we find that only dissatisfied agents move across the matchings in an absorbing set, while satisfied ones remain with the same partner permanently (Theorem 2). In addition, we prove that if a roommate problem has multiple absorbing sets there are similarities between their corresponding matchings (Theorem 3). Specifically, any two absorbing sets differ only in how the satisfied agents are matched. We also show that if a single agent is satisfied, then it remains single in all matchings in all absorbing sets (Corollary 2). To conclude, we compare absorbing sets to other solutions concepts proposed in the literature to solve the class of roommate problems with strict preferences. We also discuss the possibility of extending our results to more general models.

The rest of the paper is organized as follows: Section 2 contains the preliminaries. In Section 3 we study absorbing sets for a roommate problem. Section 4 characterizes these sets. In Section 5 we study the structure of the matchings in absorbing sets. Some concluding remarks are contained in Section 6. Appendix A presents an iterative process for finding the stable partitions that determine absorbing sets. Appendix B is divided into two parts, one with the lemmas used in the paper and their proofs and the other with the proofs of the theorems, propositions and corollaries in the text.

## 2. Preliminaries

A *roommate problem* or simply, when there is no risk of confusion, a *problem* is a pair $(N, (\succcurlyeq_x)_{x \in N})$ where $N$ is a finite set of agents and for each agent $x \in N$, $\succcurlyeq_x$ is a complete, transitive preference relation defined over $N$. Let $\succ_x$ be the strict preference associated with $\succcurlyeq_x$. In this paper we analyze problems with strict preferences, which we denote by $(N, (\succ_x)_{x \in N})$.

[3] As far as we know, the notion of absorbing sets applied as a solution to abstract systems was first introduced by Schwartz (1970). It coincides with the elementary dynamic solution (Shenoy, 1979, 1980). The union of absorbing sets gives the admissible set (Kalai et al., 1976). We have studied absorbing sets instead of the admissible set because they have interesting structural properties.

[4] A matching "dominates" another matching if it can be obtained by a finite sequence of matchings such that each matching in this sequence directly dominates the previous one.

[5] Gusfield and Irving (1989) raise an open question about the possibility of finding a condition for verifying the stability of a roommate problem. Tan (1991) answers the question in the affirmative by introducing the notion of stable partitions.