Games and Economic Behavior 76 (2012) 340–343

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

A correction to "Large games and the law of large numbers" [Games Econom. Behav. 64 (2008) 1–34] ☆

Juha Tolvanen [a,b,*], Elefterios Soultanis [c]

[a] Helsinki Institute for Information Technology HIIT, P.O. Box 19215, Aalto, Finland
[b] Princeton University, Department of Economics, Fisher Hall, Princeton, NJ 08544-1021, USA
[c] University of Helsinki, P.O. Box 68, FI-00014 Helsinki, Finland

ARTICLE INFO

Article history:
Received 30 May 2010
Available online 26 June 2012

JEL classification:
C7
C6
D84

Keywords:
Large games
Equilibria
Law of large numbers

ABSTRACT

Nabil Al-Najjar (2008) showed how games with countably infinite player sets can be used to approximate games with large finite player sets. Unfortunately, we have found an error in the proof of Al-Najjar's Theorem 5. In this correction we discuss the error and offer two slightly weaker versions of the theorem.

1. Introduction

Al-Najjar (2008) constructs a framework for modeling large anonymous games. The model offers one, fairly simple solution for the technical measurability problems that have pestered earlier continuum-player models, such as the pioneering work by Schmeidler (1973).

However, his Theorem 5 has a considerably stronger statement compared to the proof offered. In this corrigendum we first illustrate this error by giving a simple counterexample and then propose two somewhat weaker versions of the original theorem. Throughout the paper we use Al-Najjar's notation where applicable and define only any new concepts we use.

2. Alternative formulations for Al-Najjar's Theorem 5

In his Theorem 5, Al-Najjar states the following: "Suppose that Γ represents a proper sequence of finite-player games $\{\Gamma_N\}_{N=1}^{\infty}$. If for each N, μ_N is an ε_N-equilibrium for Γ_N with $\varepsilon_N \downarrow 0$, then there is an equilibrium μ for Γ and a subsequence $\{N_k\}$ such that $\mu_{N_k}(t) \to \mu(t)$ for every $t \in T$." In his proof, he picks a subsequence $\{N_k\}_{k\in\mathbb{N}}$ satisfying

$$\lim_{k\to\infty} \int \mu_{N_k}\, \mathrm{d}\lambda_{N_k} = \mathcal{U} - \lim_{N\to\infty} \int \mu_N\, \mathrm{d}\lambda_N.$$

DOI of original article: http://dx.doi.org/10.1016/j.geb.2007.11.002.
☆ The authors would like to thank the anonymous referee and Faruk Gul for some helpful comments.
* Corresponding author at: Helsinki Institute for Information Technology HIIT, P.O. Box 19215, Aalto, Finland.
E-mail addresses: juha.k.tolvanen@gmail.com (J. Tolvanen), elefterios.soultanis@helsinki.fi (E. Soultanis).

0899-8256/$ – see front matter
http://dx.doi.org/10.1016/j.geb.2012.05.010

Then for every $t \in T$ he defines $N_k(t) = \min\{N_k:\ k \in \mathbb{N},\ t \in T_{N_k}\}$ i.e. $\Gamma_{N_k(t)}$ is defined as the first game in the subsequence $(\Gamma_{N_k})_{k\in\mathbb{N}}$ where player t appears. After that he continues to prove that the profile defined by $\mu(t) \equiv \mu_{N_k(t)}(t)$ is an equilibrium. It is our understanding that the proof holds in the sense that μ is really an equilibrium. However, the following example illustrates how that equilibrium need not be the pointwise limit of $(\mu_{N_k}(t))_{k\in\mathbb{N}}$ for *any* $t \in T$.

Example 1. Take a proper sequence of finite-player games where $A = \{e_1, e_2\}$, the set of standard base vectors of $\mathbb{R}^2$, and define the utility functions by the formula:

$$u_N\big(t, a, (\delta, 1-\delta)\big) = \min\{2\delta, 2-2\delta\}$$

for every $\delta \in [0,1]$, $t \in T_N$, $a \in A$ and $N \in \mathbb{N}$. Furthermore, suppose that every $\#T_N$ is even. Enumerate the players so that $T_N = \{t_1, \dots, t_{\#T_N}\}$ and do not change the enumeration of the players from the previous player sets when more players are added. Consider then the following strategy profiles: For every $N \in \mathbb{N}$, let

$$\mu_N(t_n) = \left(\frac{1}{2} + \frac{1}{2N}, \frac{1}{2} - \frac{1}{2N}\right), \quad \text{if } n \text{ is even}$$

and

$$\mu_N(t_n) = \left(\frac{1}{2} - \frac{1}{2N}, \frac{1}{2} + \frac{1}{2N}\right), \quad \text{if } n \text{ is odd.}$$

Since $\#T_N$ is even, the expected proportion of players playing e_1 is $\frac{1}{2}$ for every N. This maximizes the utility function for all of the players and hence the profile μ_N must be an equilibrium.

Note that $\lim_{N\to\infty} \mu_N(t_n) = (\frac{1}{2}, \frac{1}{2})$, for any $n \in \mathbb{N}$. Now take an *arbitrary* subsequence of $(\mu_N)_{N=1}^{\infty}$, say $(\mu_{N_k})_{k=1}^{\infty}$. As every subsequence of a convergent sequence converges to the same limit, we know that $\lim_{k\to\infty} \mu_{N_k}(t) = (\frac{1}{2}, \frac{1}{2})$. Following Al-Najjar's proof define $\mu(t) \equiv \mu_{N_k(t)}(t)$. This then means that for any $n \in \mathbb{N}$, $\mu(t_n) = \mu_{N_k(t_n)}(t_n) = (\frac{1}{2} + (-1)^n \frac{1}{2N_k(t)}, \frac{1}{2} - (-1)^n \frac{1}{2N_k(t)}) \neq (\frac{1}{2}, \frac{1}{2})$. Thus the profile suggested in Al-Najjar's proof as the limiting equilibrium is not a pointwise limit of any subsequence of the equilibria $(\mu_N)_{N=1}^{\infty}$ for any player $t \in T$.

While the claim of pointwise convergence in Al-Najjar's Theorem 5 is erroneous, there are at least two ways to weaken its statement to get the theorem to hold: One can either choose a weaker form of convergence or add additional assumptions. Theorem 5′ below keeps the original assumptions but yields only a weak analogue of L_1-convergence between the sequence of finite-player profiles and the discrete large game profile. The limiting equilibrium will then be unique only in this L_1-sense. To save the uniqueness of the limit in Al-Najjar's original theorem, one could keep the theorem as it is but add the assumption that there exists a profile μ, for which $\lim_{N\to\infty} \sup_{t\in T_N} |\mu_N(t) - \mu(t)| = 0$. In this case it can be shown that μ is not merely an equilibrium but an exact equilibrium of the limiting discrete large game. Proof of this result is available from the authors on request. The formal version of the theorem with Al-Najjar's original assumptions is the following[1]:

Theorem 5′. *Suppose that Γ represents a proper sequence of finite-player games $\{\Gamma_N\}_{N=1}^{\infty}$. If for each N, μ_N is an ε_N-equilibrium for Γ_N with $\varepsilon_N \downarrow 0$, then there exists an equilibrium profile $\mu(t)$ for Γ which satisfies*

$$\lim_{N\to\infty} \sum_{t\in T_N} \frac{|\mu(t) - \mu_N(t)|}{\#T_N} = \lim_{N\to\infty} \int_{T_N} \big|\mu(t) - \mu_N(t)\big|\, \mathrm{d}\lambda_N = 0. \tag{2.1}$$

Note that compared to the original version, in the corrected version there is no need to move to a subsequence of strategy profiles.

Proof of Theorem 5′. Let $\hat{N}(t)$ be the smallest N for which $t \in T_N$. For all $t \in T$ define $\mu(t) = \mu_{\hat{N}(t)}(t)$. Then the following estimate holds:

$$\begin{aligned}
\int_{T_N} |\mu - \mu_N|\, \mathrm{d}\lambda_N &= \frac{1}{\#T_N} \sum_{t\in T_N} \big|\mu(t) - \mu_N(t)\big| \\
&= \frac{1}{\#T_N} \sum_{t\in T_{N-1}} \big|\mu(t) - \mu_N(t)\big| + \frac{1}{\#T_N} \sum_{t\in T_N\setminus T_{N-1}} \big|\mu(t) - \mu_N(t)\big| \\
&\leqslant \frac{\#T_{N-1}}{\#T_N} + \frac{1}{\#T_N} \sum_{t\in T_N\setminus T_{N-1}} \big|\mu(t) - \mu_N(t)\big|.
\end{aligned}$$

[1] We would like to thank the anonymous referee for pointing out this version of the theorem.