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Dynamics in tree formation games

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ABSTRACT

Network formation games capture two conflicting objectives of selfish nodes in a network: such nodes wish to form a well-connected network and, at the same time, to minimize their cost of participation. We consider three families of such models where nodes avoid forming edges beyond those necessary for connectivity, thus forming tree networks. We focus on two *local two-stage* best-response dynamics in these models, where nodes can only form links with others in a restricted neighborhood. Despite this locality, both our dynamics converge to efficient outcomes in two of the considered families of models. In the third family of models, both our dynamics guarantee at most constant efficiency loss. This is in contrast with the standard best-response dynamics whose efficiency loss is unbounded in all three families of models. Thus we present a globally constrained network formation game where local dynamics naturally select desirable outcomes.

1. Introduction

Network formation games (Jackson, 2008) typically embody a tension between two countervailing forces. On one hand, individuals benefit from connections to each other because of the network reachability such connections afford; for example, in a transportation network, additional links allow a node to shorten the distance to other nodes in the network. On the other hand, link creation and maintenance is typically costly. Thus individuals are incentivized to balance the number of links with the benefits they provide.

In this paper, we study this particular tension in the context of a network formation game model inspired by data communication and transportation networks. Our cost structure has two key features. First, in these applications, connectivity is typically a minimal requirement. To account for the desire for connectivity, we assume that nodes experience a very large cost when the network is disconnected. Second, we assume that in a connected network, nodes experience a cost consisting of two terms: (1) a *link maintenance cost*, $\beta \geqslant 0$, per link the node participates in; and (2) a *traffic cost*, per unit of traffic sent by the node. We assume that nodes have an ex ante demand to send traffic to each other and form links with each other with the goal of minimizing the total cost of both link maintenance and traffic handling.

In our model the entire cost of sending traffic is borne by the sender of the traffic itself. This is the case, for example, in logistics networks, where transportation costs are borne by the sender. It is also the case in certain types of data transport contracts for Internet service. In our paper we consider several forms for the traffic cost; special cases of this cost recover

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both the model of Corbo and Parkes (2005) and the connections model of Jackson and Wolinsky (1996). (Indeed, the model where the sender bears the cost of traffic is one special case of an allocation rule, as defined by Jackson, 2008.)

Our model has two orthogonal dimensions that can be explored. We can fix the value of the link maintenance cost β and study the static game and dynamics for a range of suitable traffic-related cost functions. Alternatively, we can fix the traffic-related cost function and vary the link maintenance cost β. We select the former approach and fix β to be large enough so that the cost of redundant links is prohibitive. This large value of β together with the requirement of connectivity of the network implies that the set of pairwise Nash networks is a set of trees.

Our work is primarily concerned with dynamics: in particular, are there plausible dynamics that lead to efficient or near-efficient equilibria? We begin by analyzing a natural starting point, myopic best-response dynamics. First, we characterize equilibrium outcomes of the static game and show that even the *strongly stable* outcomes may be inefficient. In fact, the corresponding price of anarchy can grow *linearly* in the size of the network. Second, we observe that any static equilibria is a fixed point of best-response dynamics. Hence, even with respect to strong stability, the best-response dynamics may select a grossly inefficient network.

This inefficiency problem is addressed by considering two particular dynamic processes inspired by our earlier work (Arcaute et al., 2007, 2008, 2009), that we call NodeDynamics and LinkDynamics. In NodeDynamics, at the beginning of a round, an *active* node u is exogenously selected by a stochastic process. Node u is allowed to minimize the cost it experiences at the end of the round by undertaking either a unilateral deviation or a bilateral deviation with any node w in the active node's vicinity. In case of unilateral deviation, the active node is allowed to erase several of its adjacent edges. A bilateral deviation yields erasure of several links adjacent to either u or w and creation of a new link between these two nodes. This deviation proceeds in two stages. At the first stage, the active node u erases some of its adjacent links and suggests node w to create a link between them. At the second stage, node w erases some of its adjacent links and decides whether to accept the link proposal from node u. We assume that link proposal is accepted if the cost experienced at the end of the round by node w is smaller than the one at *the end of the first stage*.

In contrast, in LinkDynamics, at the beginning of a round, a pair of nodes is exogenously selected by a stochastic process and one of the nodes is designated to be an active node. We let u denote the active node and let w denote the other node. We assume that the dynamics are local, i.e., the selected nodes u and w are necessarily close to each other in the current network. Node u is allowed to minimize the cost it experiences at the end of the round by undertaking either a unilateral deviation or a bilateral deviation with *the selected* node w. Unilateral and bilateral deviation are performed in the identical manner to these in NodeDynamics. Essentially, the activation process in LinkDynamics selects a potential link (a pair of nodes) to activate as opposed to NodeDynamics in which a single node is activated.

We analyze NodeDynamics and LinkDynamics for three specific subclasses of the distance-based traffic cost models. We show that our dynamics converge almost surely in all three subclasses. We then show that in two subclasses of models, these dynamics converge to the most efficient network. In the third subclass of models, the ratio of efficiency of the selected outcome to the most efficient network is shown to be bounded by a constant that is independent of the size of the network. We show that the expected time until convergence of our dynamics is polynomial for two subclasses of models and, in the third subclass, it is at most exponential. Any fixed point of NodeDynamics and LinkDynamics is necessarily a 2-stable equilibrium, therefore it is fair to compare the efficiency of networks selected by these dynamics to the efficiency of the standard best-response dynamics with respect to 2-stability. Recall that the price of anarchy associated with strong stability is of $O(|V|)$. This implies that the price of anarchy of the standard best response dynamics with respect to 2-stability is at least of $O(|V|)$. In all three subclasses, it can be seen that the outcomes selected by our dynamics are significantly more efficient than the worst case 2-stable equilibrium.

Our analysis of NodeDynamics and LinkDynamics relies on an auxiliary *tree formation game*. This game does not have a value of its own, however, it helps us to streamline the analysis of NodeDynamics and LinkDynamics in the original game. Tree formation game is a version of Myerson link announcement game where:

- all nodes incur an unbounded cost in any outcome other than a tree;
- the link maintenance cost is removed and the traffic-related cost retained; and
- any deviation involving one or two nodes is allowed.

We define two constrained versions of best-response dynamics of the tree formation game, dubbed TreeNode and TreeLink. We show that TreeNode is equivalent to NodeDynamics in the original game and that TreeLink is equivalent to LinkDynamics. We derive convergence results for TreeNode and TreeLink, as they are easier to analyze. Due to the equivalence of the dynamics, these results automatically apply to NodeDynamics and LinkDynamics.

The rest of the paper is organized as follows. We begin with a survey of relevant literature in Section 2. Section 3 presents the formal definition of our link announcement game in its most general form. This section also introduces the corresponding notions of equilibrium and efficiency and the results of our analysis of the static game. In Section 4 we formally define NodeDynamics and LinkDynamics and analyze some of their basic properties. In Section 5 we introduce distance-based cost models, and we constrain our analysis to these cost models for the remainder of the paper. In Section 6 we introduce the tree formation game and the two versions of the associated best-response dynamics TreeNode and TreeLink. We show the equivalence of the latter to NodeDynamics and LinkDynamics, respectively. In Sections 7 and 8 we address three specific subclasses of the distance-based cost model. For each class, we refine our characterization of stable outcomes of the link