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Note

On the folk theorem with one-dimensional payoffs and different discount factors

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ABSTRACT

Proving the folk theorem in a game with three or more players usually requires imposing restrictions on the dimensionality of the stage-game payoffs. Fudenberg and Maskin (1986) assume full dimensionality of payoffs, while Abreu et al. (1994) assume the weaker NEU condition ("nonequivalent utilities"). In this note, we consider a class of n-player games where each player receives the same stage-game payoff, either zero or one. The stage-game payoffs therefore constitute a one-dimensional set, violating NEU. We show that if all players have different discount factors, then for discount factors sufficiently close to one, any strictly individually rational payoff profile can be obtained as the outcome of a subgame-perfect equilibrium with public correlation.

1. Introduction

For the folk theorem to hold with more than two players, it is necessary to have the ability to threaten any single player with a low payoff, while also offering rewards to the punishing players. In assuming full dimensionality of the convex hull of the set of feasible stage-game payoffs, Fudenberg and Maskin (1986) guarantee that those individual punishments and rewards exist. Abreu et al. (1994) show that the weaker NEU condition ("nonequivalent utilities"), whereby no two players have identical preferences in the stage game, is sufficient for the folk theorem to hold.

When the NEU condition fails, players that have equivalent utilities can no longer be individually punished in equilibrium. Wen (1994) introduces the notion of *effective minmax* payoff, which takes into account the fact that when a player is being minmaxed, another player with equivalent utility might unilaterally deviate and best respond. The effective minmax payoff of a player cannot be lower than his individual minmax payoff (when NEU is satisfied, they coincide), and Wen shows that when NEU fails it is the effective minmax that constitutes the lower bound on subgame-perfect equilibrium payoffs. He establishes the following folk theorem: when players are sufficiently patient, any feasible payoff vector can be supported as a subgame-perfect equilibrium, provided it dominates the effective minmax payoff vector. This could be relaxed by allowing for unequal discounting.

As pointed out by Lehrer and Pauzner (1999), when players have different discount factors, the set of feasible payoffs in a two-player repeated game is typically larger and of higher dimensionality than the set of feasible stage-game payoffs.[1] In a particular three-player game in which two players have equivalent utilities, Chen (2008) illustrates how with unequal discounting payoffs below the effective minmax may indeed be achieved in equilibrium for one of the players.

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[1] See also Mailath and Samuelson (2006, Remark 2.1.4) for a simple illustration of how players can trade payoffs over time.

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C	L	R
T	1,1,1	0,0,0
B	0,0,0	0,0,0

D	L	R
T	0,0,0	0,0,0
B	0,0,0	1,1,1

Fig. 1. A stage game with one-dimensional payoffs.

In this note, we explore the notion that unequal discounting restores the ability to punish players individually in an n-player game where all players have equivalent utilities. Our result is stronger than Chen's as we show that all players can be hold down to their individual minmax payoff in equilibrium. Moreover we argue that our result holds for all possible violations of NEU. We find that a small difference in the discount factors suffices to hold a player to his individual minmax for a certain number of periods while still being able to reward the punishing players. For discount factors sufficiently close to one, any strictly individually rational payoff, including those dominated by the effective minmax payoff, can be obtained as the outcome of a subgame-perfect equilibrium with public correlation, restoring the validity of the folk theorem.

Although our result is stated for games where all players have equivalent utilities, we conjecture that it extends to weaker violations of NEU, as long as any two players with equivalent utilities have different discount factor. The intuition behind this conjecture is that following Abreu et al. (1994) we could design specific punishments for each group of players with equivalent utilities and use the difference in discount factors within each group to enforce those specific punishments.

1.1. An example

Consider the stage game in Fig. 1, where Player 1 chooses rows, Player 2 columns and Player 3 matrices. This stage game is infinitely repeated and the players evaluate payoff streams according to the discounting criterion. When the players share a common discount factor $\delta < 1$, Fudenberg and Maskin (1986, Example 3) show that any subgame-perfect equilibrium yields a payoff of at least $1/4$ (the effective minmax) to each player, whereas the individual minmax payoff of each player is zero.[2] The low dimensionality of the set of stage-game payoffs weakens the punishment that can be imposed on a player as another player with equivalent utility can deviate and best respond. The inability to achieve subgame-perfect equilibrium payoffs in $(0, 1/4)$ means that the "standard" folk theorem fails in this case.

We show however that if all three players have different discount factors, there exists a subgame-perfect equilibrium in which the payoff to each player is arbitrarily close to zero, the individual minmax, provided that the discount factors are sufficiently close to one. Any payoff in the interval $(0, 1/4)$ can then be achieved in equilibrium, restoring the validity of the folk theorem in the context of this game.

1.2. Notation

We consider an n-player repeated game, where all players have equivalent utilities. We normalize payoffs to be in $\{0, 1\}$ and let each player's individual minmax payoff be zero.[3] We use public correlation to convexify the payoff set, although we argue later that this assumption can be dispensed with. Players have different discount factors, and are ordered according to their patience level: $0 < \delta_1 < \cdots < \delta_{n-1} < \delta_n < 1$.[4] We use an exponential representation of discount factors: $\forall i$, $\delta_i := e^{-\Delta\rho_i}$, where $\Delta > 0$ could represent the length of time between two repetitions of the stage game. As $\Delta \to 0$, all discount factors tend to one. The ρ's are strictly ordered: $0 < \rho_n < \cdots < \rho_2 < \rho_1$. We assume that the stage game has a (mixed) Nash equilibrium which yields a payoff $Q < 1$ to all players.[5]

We summarize our assumptions about the game and introduce a notation for the lowest subgame-perfect equilibrium payoff of a player i in the following definitions:

Definition 1. Let $\Gamma(\Delta)$ be the set of n-player infinitely repeated games such that:

A1. The set of stage-game payoffs is one-dimensional and all players receive the same payoff in $\{0, 1\}$.
A2. The stage game has a mixed-strategy Nash equilibrium which yields a payoff of $Q < 1$ to all players.
A3. Each player's pure action individual minmax payoff is zero.
A4. Players evaluate payoff streams according to the discounting criterion, and discount factors are strictly ordered: $0 < \delta_1 < \cdots < \delta_n < 1$, where $\delta_i := e^{-\Delta\rho_i}$.

Note that the stage game of Fig. 1 satisfies assumptions A1 to A3 of Definition 1.

[2] For example, when Player 1 plays T and Player 2 plays R, Player 3 gets a payoff of 0 whether he plays C or D.
[3] We only use two payoffs as we only need to consider the minmax payoff and the maximum possible payoff.
[4] Note that the result no longer holds if several players have the same discount factor. We address this point before Section 2. We thank a referee for suggesting clarification on that point.
[5] For example in the game of Fig. 1, the mixture $\{(1/2, 1/2), (1/2, 1/2), (1/2, 1/2)\}$ is a Nash equilibrium that yields a payoff of $1/4$.