Games and Economic Behavior 71 (2011) 487–502

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Equilibrium play in matches: Binary Markov games ☆

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A R T I C L E   I N F O

Article history:
Received 30 May 2009
Available online 12 May 2010

JEL classification:
C72
C73

Keywords:
Stochastic games
Minimax
Strictly competitive games
Game theory and sports
Tennis

A B S T R A C T

We study two-person extensive form games, or "matches," in which the only possible outcomes (if the game terminates) are that one player or the other is declared the winner. The winner of the match is determined by the winning of points, in "point games." We call these matches *binary Markov games*. We show that if a simple monotonicity condition is satisfied, then (a) it is a Nash equilibrium of the match for the players, at each point, to play a Nash equilibrium of the point game; (b) it is a minimax behavior strategy in the match for a player to play minimax in each point game; and (c) when the point games all have unique Nash equilibria, the only Nash equilibrium of the binary Markov game consists of minimax play at each point. An application to tennis is provided.

1. Introduction

In many games, players are advised to play the same way regardless of the score. In tennis, for example, players are often advised by their coaches to play each point the same, whether the point is a match point or the opening point of the match. In poker it is a common belief that a player should play the same whether he is winning or losing.[1] This notion that one should play in a way that ignores the score stands in sharp contrast to the alternative view, also commonly espoused, that one should play differently on "big points," *i.e.*, in situations that are "more important." We establish in this paper that for a certain class of games, which we call *binary Markov games*, equilibrium play is indeed independent of the game's score.

We consider two-player games — we call them "matches" — which are composed of points, and in which (a) the history of points won can be summarized by a score, or state variable; (b) when the players compete with one another to win points, they do so via a "point game" which may depend upon the current score or state of the match; and (c) each player cares only about whether he wins or loses the match — *i.e.*, about whether the match terminates in a winning state for himself (and thus a loss for his opponent), or vice versa. We use the terminology "match" and "point game" in order to distinguish the overall game from the games in which the players compete for points.

Many real-life games, such as tennis, fit the description we have just given. It is useful, however, to begin by describing a much simpler example: two players play "matching pennies" repeatedly against one another; the winner of each matching pennies game wins a point; and the first player to be ahead by two points wins the overall game, or match. This game has only five potential scores, −2, −1, 0, +1, and +2, where the score +2 means Player A is ahead by two points (and has thus

☆ This research was partially supported by NSF grant #SBR-0099353. We are grateful to three referees and an advisory editor for detailed and insightful comments.

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[1] But only in cash games, not in tournaments.

0899-8256/$ – see front matter
doi:10.1016/j.geb.2010.04.011

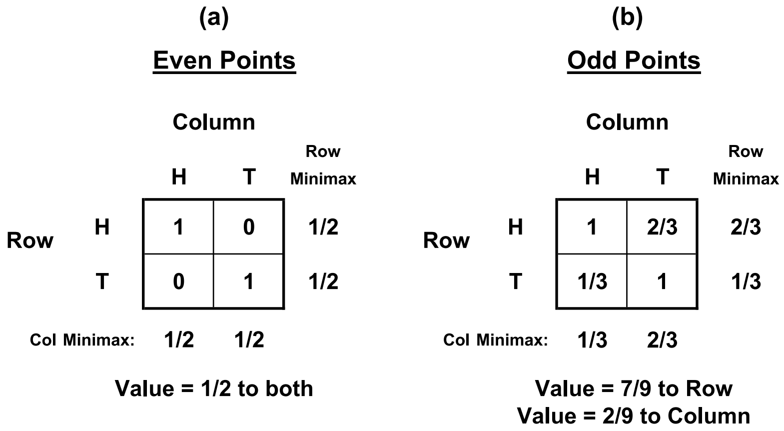

Fig. 1. Outcomes (cell entries) are the probability that row wins the point.

won the game, *i.e.*, the match), the score -2 means Player A has lost the match, the score 0 means the match is tied (this is the score when the match begins), and the scores $+1$ and -1 mean, respectively, that Player A is ahead by one point and that Player A is behind by one point. In this game, like all the games we consider, the players are interested in winning points only as a means to winning the match.

Now let's change the example slightly. Suppose that, as before, when the score is tied (*i.e.*, when the match is in state 0) the players play the conventional matching pennies game: let's say Row wins the point when the coins match, and Column wins the point when the coins don't match, as depicted in Fig. 1a. But when the score is not tied, and the match has not yet ended (*i.e.*, when the state is "odd," either $+1$ or -1), a slightly different matching-pennies game determines which player wins the current point, namely the game depicted in Fig. 1b. In this game the players still choose Heads or Tails, and Row still wins if their coins match. But if the coins don't match, then Nature randomly determines which player wins the point, and the player who chose Heads has a $2/3$ chance of winning, the player who chose Tails only a $1/3$ chance.

There are several things worth noting in this new match. First, just as in the conventional matching pennies game, the new "odd-state" point game has a unique equilibrium, which is in mixed strategies. But the equilibrium (and minimax) mixtures are different in the odd-state point game than in the conventional game: here the Row player plays Heads with mixture probability $2/3$ instead of $1/2$, and the Column player plays Heads with mixture probability $1/3$. The value of the odd-state point game to the Row player is $7/9$ (Row's probability of winning the point), and the game's value to the Column player is $2/9$. Thus, by always playing his minimax mixture in the current point game, the Row player can assure himself a probability $1/2$ of winning any point played when the score is even, and a $7/9$ probability of winning any point played when the score is not even. Similarly, the Column player, by always playing his point-game minimax strategy, can assure himself a $1/2$ probability of winning even points and a $2/9$ probability of winning odd points.

It is easy to verify that if the players always play their minimax mixtures in each point game, then the Row player will win the match with probability $7/9$ if the match is tied, with probability $77/81$ if he is ahead by a point, and with probability $49/81$ if he is behind by a point. Indeed, the Row player can assure himself of at least these probabilities of winning, no matter what his opponent does, by always playing his minimax mixture, and the parallel statement can be made for the Column player. Note that this "minimax play" prescription says that one's play should not depend on the match's score, except to the extent that the point game depends on the score: each player should play the same whether he is ahead in the match or behind (*viz.*, Row should mix $2/3$ on Heads, Column should mix $1/3$ on Heads).

It seems clear, at least intuitively, that such play is a Nash equilibrium in the match, and that a minimax strategy for playing the match is to play one's minimax mixture in every state of the match — and perhaps even that this is the *only* equilibrium in the match. Proving these propositions in general for binary Markov games, however, turns out to be non-trivial. This is largely because the match is not a game with a finite horizon: for example, if neither player in the example we have just described is ever ahead by two points, the match continues indefinitely.

We will identify a general class of games like the one above, in which we will show that equilibrium play at any moment is dictated only by the point game currently being played — play is otherwise independent of the history of points that have been won or actions that have been taken, and in particular it is independent of the score, except to the extent that the current point game depends on the score.

The results we obtain are for matches in which every point game is a strictly competitive win–loss game (*i.e.*, in each point game either one player or the other wins a single point).[2] Our first result, an Equilibrium Theorem, establishes that

[2] Wooders and Shachat (2001) also obtain results on equilibrium and minimax play in sequential play of stage games where at each stage one player wins and the other loses. Their stage games can thus be interpreted as contests for points. But in Wooders and Shachat (a) it is always strictly better for a player to win more points, and (b) the "match" has a known, finite length. In contrast, we assume here that (a) each player cares only about whether he wins or loses the match, and (b) the length of the match is allowed to be indefinite and infinite.