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VISIM: Sequential simulation for linear inverse problems☆

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Abstract

Linear inverse Gaussian problems are traditionally solved using least squares-based inversion. The center of the posterior Gaussian probability distribution is often chosen as the solution to such problems, while the solution is in fact the posterior Gaussian probability distribution itself. We present an algorithm, based on direct sequential simulation, which can be used to efficiently draw samples of the posterior probability distribution for linear inverse problems. There is no Gaussian restriction on the distribution in the model parameter space, as inherent in traditional least squares-based algorithms.

As data for linear inverse problems can be seen as weighed linear averages over some volume, block kriging can be used to perform both estimation (i.e. finding the center of the posterior Gaussian pdf) and simulation (drawing samples of the posterior Gaussian pdf). We present the kriging system which we use to implement a flexible GSLIB-based algorithm for solving linear inverse problems.

We show how we implement such a simulation program conditioned to linear average data. The program is called VISIM as an acronym for Volume average Integration SIMulation. An effort has been made to make the program efficient, even for larger scale problems, and the computational efficiency and accuracy of the code is investigated.

Using a synthetic cross-borehole tomography case study, we show how the program can be used to generate realizations of the a posteriori distributions (i.e. solutions) from a linear tomography problem. Both Gaussian and non-Gaussian a priori model parameter distributions are considered.

Keywords: Kriging; Sequential simulation; Linear inverse theory

1. Introduction

Some data **d** are indirect measurements, and hence a function of some model parameters **m** (typically describing subsurface structure). Let the forward problem be the problem of calculating data from a given set of model parameters using a function **g**, typically related to some physical problem, such that $\mathbf{d} = \mathbf{g}(\mathbf{m})$. We refer to the problem of inferring properties of **m** from measurements **d** as the inverse problem.

In a Bayesian formulation the solution to an inverse problem is a probability density function (pdf) (the posterior pdf, σ) which is given as a normalized (K is a normalization factor) product of the pdf describing the a priori information (the prior pdf, ρ)

☆Code is available from server at http://www.iamg.org/CGEditor/index.htm.

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and the likelihood function (Δ), related to the forward operator $\mathbf{g}$ (which is related to some physical law) (Tarantola, 2005):

$$\sigma(\mathbf{d}, \mathbf{m}) = K\rho(\mathbf{d}, \mathbf{m})\Delta(\mathbf{d}, \mathbf{m}). \tag{1}$$

In case the inverse problem is linear, such that the operator $\mathbf{g}$ is linear, and Gaussian, such that the prior pdf and the noise distribution is Gaussian, the solution to the inverse problems is a Gaussian pdf described by a mean and a covariance.

The center of this Gaussian a posteriori pdf is often chosen as the 'solution' to the inverse problem, but this point cannot adequately describe the a posteriori pdf. To do this a representative sample (a set of realizations) of the posteriori pdf must be generated, from which a posteriori statistics can be obtained. This can be done using for example Markov chain Monte Carlo methods (Mosegaard and Tarantola, 1995). This is, however, computationally very expensive.

Hansen et al. (2006) show how the a posteriori Gaussian distribution can be exactly described by a simple kriging system with noisy data of mixed support (that is, point- as well as volume support). Therefore sequential simulation techniques can be used to efficiently draw samples of the a posteriori pdf, as described by Hansen et al. (2006) and Gómez-Hernández and Cassiraga (2000).

In this manuscript we will introduce the kriging system to deal with noisy data of mixed support. Using this kriging system we show how to implement a sequential simulation program that generates realizations of a random field with a chosen a priori mean, variance, covariance and histogram, honoring observations of point- as well as volume support.

This approach can be used to solve linear inverse problems with an a priori two-point covariance model, in the sense that actual samples from the a posteriori distribution of a linear inverse problem can be drawn in a computationally efficient manner. This is an improvement over traditional least squares-based linear inversion where only the center of the Gaussian posterior pdf is chosen as the solution.

Making use of direct sequential simulation (dssim), the method we propose is not restricted to the assumption of a Gaussian distribution over the model parameter space as in traditional least squares-based linear inversion. Any histogram can be used to describe the prior distribution over the model parameter space.

Using a synthetic cross-borehole tomography example we will illustrate the use of the program as well as investigate the computational efficiency and accuracy of a number of available simulation and estimation options.

2. Theory

Hansen et al. (2006) describe how samples of the a posteriori pdf of linear Gaussian inverse problems can be generated using sequential simulation. The least squares system used by Hansen et al. (2006) as part of sequential simulation, can be formulated as a simple kriging system with weighed linear average data with associated Gaussian measurement error. This is the kriging system we use as part of the presented sequential simulation algorithm, and it will be briefly introduced here.

Let $Z(\mathbf{u})$ denote the measurement of some random variable Z at the location $\mathbf{u}$ with measurement error R. This is the definition of a datum of 'point support'. For the remainder of this paper we will refer to measurement errors as errors that are non-systematic, uncorrelated with the random function Z and possibly correlated among themselves, following Chiles and Delfiner (1999, p. 211).

Let Z_v be the measurement of a weighted linear average of Z over the block v, with measurement error R, such that

$$Z_v = \frac{1}{|v|}\int_v w(\mathbf{u})Z(\mathbf{u})\,\mathrm{d}\mathbf{u} + R \tag{2}$$

$$\approx \frac{1}{N\sum w(u_i)}\sum_{i=1}^{N} w(u_i)Z(u_i) + R, \tag{3}$$

where $w(\mathbf{u})$ is an averaging kernel allowing variable weight within the defined support. $|v|$ is the volume of the support. This is the definition of a datum with 'volume support'. Any datum of a Gaussian linear inverse problem can be given by Z_v.

In the discrete case, Eq. (3), $w(u_i)$ are averaging weights for each of the N points, $\mathbf{u} = [u_1, u_2, \ldots, u_N]$ in the support v. If $N = 1$ and $w(u_1) = 1$, Eq. (3) reduces to an expression for a measurement of point support, and therefore data of any support is described by Eqs. (2)–(3).

Given n data, $Z_{v,i}$ where $i = 1, \ldots, n$, interpreted as average measurements of a realization of a stationary random function, RF, and an a priori model of the mean m, variance σ_0^2, and covariance C (two-point spatial connectivity) of the RF, the mean and variance of the Gaussian pdf, $N(\mu_k, \sigma_k^2)$,