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Stackelberg thresholds in network routing games or the value of altruism ☆

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ABSTRACT

It is well known that the Nash equilibrium in *network routing games* can have strictly higher cost than the optimum cost. In Stackelberg routing games, where a fraction of flow is centrally-controlled, a natural problem is to route the centrally-controlled flow such that the overall cost of the resulting equilibrium is minimized.

We consider the scenario where the network administrator wants to know the minimum amount of centrally-controlled flow such that the cost of the resulting equilibrium solution is *strictly* less than the cost of the Nash equilibrium. We call this threshold the *Stackelberg threshold* and prove that for networks of parallel links with linear latency functions, it is equal to the minimum of the Nash flows on links carrying more optimum flow than Nash flow.

Our approach also provides a simpler proof of characterization of the minimum fraction that must be centrally controlled to induce the optimum solution.

1. Introduction

Noncooperative network routing games are a nice model of the behavior of selfish users trying to optimize their own benefit. In such a game, each player intends to send a fixed amount of flow from its source to its sink using a shortest delay path through the given network in a noncooperative manner.

The solution reached by players selfishly routing their flow is called the Nash equilibrium or Nash flow. Since players choose their paths to minimize their own delay alone, the quality of the resulting Nash equilibrium in general may be worse than the quality of the optimum way to route flow through the network so as to minimize the total overall latency of all users, which may be thought of as the social cost of the routing. A classic example of Pigou (1920) shows that this can indeed be the case. The ratio of the cost of the Nash equilibrium to the optimum solution is called the *Price of Anarchy* (Koutsoupias and Papadimitriou, 1999). The idea of bounding the price of anarchy in network routing games has become well-studied after the groundbreaking work of Roughgarden and Tardos (2002). Roughgarden and Tardos show that for general latency functions, the price of anarchy can be arbitrarily large. For the class of networks with linear latency functions, however, they prove that the price of anarchy is bounded by 4/3.

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The Nash equilibrium is an attractive concept from the point of view of the study of stable equilibria since no player has any incentive to unilaterally change his/her strategy. But its inefficiency (that is, its potentially large cost compared to the social optimum) has always been a concern. There has been substantial work on ways to address this issue. Some such methods are: (i) *Mechanism design and taxes and tolls*, in which the rules of the game are established (e.g. by putting taxes and tolls on network links) to help ensure that the quality of the resulting Nash compatible with the rules is good compared to the social optimum (see, for example, Nisan and Ronen, 2001; Nisan, 1999; Beckmann et al., 1956; Cole et al., 2006; Fleischer, 2005; Feigenbaum et al., 2001), (ii) *Designing the network* in such a way that the network has good Nash to optimum ratio to start with (see, for example, Roughgarden, 2006a; Korilis et al., 1997), and (iii) *Capacity augmentation*, such that the cost of Nash equilibrium in augmented network is good compared to the cost of optimum in the original network (see, for example, Roughgarden and Tardos, 2002). See also Roughgarden's survey (Roughgarden, 2006b) for a discussion about coping with inefficiency of Nash equilibria. All of these methods necessitate either a change in the way game is played in the existing network or a change in the network itself.

Another way to improve the quality of the Nash equilibrium is to consider situations in which not all flow is routed selfishly. The motivation comes from considering networks where there is a mix of selfish and centrally controlled players. An example of such a network mentioned in Roughgarden's thesis (Roughgarden, 2002, Chapter 6) is that of a network where there may be two different prices. Clients paying the *premium* price get to choose their own route through the network and those paying the *bargain* price do not get a choice of routes—they are controlled centrally by the network administrator. Roughgarden (2004) considers the problem of routing a β fraction of flow centrally in such a way that if the remaining $1-\beta$ fraction chooses their own paths selfishly then the cost of the resulting solution is minimized. He calls the routing of the centrally controlled flow a *Stackelberg strategy* and the resulting equilibrium the *equilibrium induced by the strategy* with fraction β; we will refer to the latter as simply the *Stackelberg equilibrium*. He addresses the question of finding a Stackelberg strategy such that the cost of the resulting Stackelberg equilibrium is close to the social optimum. For a network of parallel links and centrally controlled $0 \leqslant \beta \leqslant 1$ fraction of flow, he gives a Stackelberg strategy such that the resulting Stackelberg equilibrium comes within a $1/\beta$ factor of the social optimum for arbitrary latencies and within a $4/(3+\beta)$ factor for linear latencies.

To state our result precisely, and give the context of our work with respect to the related work in literature, let us be a little more specific about the problem considered by Roughgarden (for a full description of the model, see Section 2). Let G be a network with two nodes $\{s, t\}$, a source s and a sink t, and k directed parallel links $\{1, 2, \ldots, k\}$ from s to t. Each edge i is equipped with a latency function $l_i(x) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ which is nonnegative, continuous, and nondecreasing. A total of unit flow is to be routed from s to t such that the total latency experienced by the whole flow is minimized. In other words, the socially optimum flow $f = (f_1, f_2, \ldots, f_k)$ is such that $\sum_{i=1}^k f_i = 1$ and $\sum_{i=1}^k f_i \cdot l_i(f_i)$ is minimized. (Note that the assumption of unit flow is without loss of generality. If the amount of total flow were equal to r, we could scale all latencies down by a factor of r, and make the total flow equal to unit amount, without changing anything essential in the problem.) A Stackelberg strategy $\bar{h}$ for a given fraction β is a flow $\bar{h} = (\bar{h}_1, \ldots, \bar{h}_k)$ such that $\sum_{i=1}^k \bar{h}_i = \beta$. Define $\tilde{l}_i(x) = l_i(x + \bar{h}_i)$ for $i = 1, \ldots, k$. Then the Stackelberg equilibrium *induced by* $\bar{h}$ is a Nash equilibrium routing $(1-\beta)$ flow in the graph G with latencies $\tilde{l}$. For $0 \leqslant \beta \leqslant 1$, let $c(G, l, \beta, \bar{h})$ be the cost of the Stackelberg equilibrium induced by $\bar{h}$, and let $c(G, l, \beta) = \min_{\bar{h}} c(G, l, \beta, \bar{h})$ be the cost of the optimum Stackelberg equilibrium with a β-fraction of centrally controlled flow. Then $c(G, l, 1)$ is the social optimum cost, and $c(G, l, 0)$ is the social cost of the Nash flow. It turns out that finding $c(G, l, \beta)$ for an arbitrary network and an arbitrary β is weakly NP-complete as proved in Roughgarden (2002, Chapter 6). Roughgarden (2004) has shown that $c(G, l, \beta) \leqslant \frac{1}{\beta} c(G, l, 1)$ for arbitrary latencies, and when the latency functions l_i are linear, then $c(G, l, \beta) \leqslant \frac{4}{3+\beta} c(G, l, 1)$. There has been a fair amount of followup work on finding good Stackelberg strategies; we discuss this work in Section 1.1.

In this paper, we study a simple but interesting question regarding Stackelberg equilibria in this setting: what fraction β of flow needs to be centrally controlled for there to be *any* improvement in the social cost whatsoever? We call this amount the *Stackelberg threshold* and denote it by $\sigma(G, l)$. To be more precise, $\sigma(G, l)$ is the minimum value of β such that $c(G, l, \beta + \varepsilon) < c(G, l, 0)$ for any $\varepsilon > 0$. In the network setting of Roughgarden, the Stackelberg threshold is the minimum fraction of bargain price users that the network administrator must get in order for the overall routing cost to be less than the cost of the Nash equilibrium in which everyone is able to route their own flows selfishly.

At first glance, it might appear that the threshold is trivially 0: that is, $c(G, l, \varepsilon) < c(G, l, 0)$ for any network G of parallel links. However, if the latency functions are such that $c(G, l, 0) = c(G, l, 1)$—that is, the Nash equilibrium happens to have optimum social cost—this is clearly false.

As this example points out, the threshold depends on the price of anarchy of the instance. This is also implied by Roughgarden's result. For linear latency functions, Roughgarden and Tardos (2002) show that $c(G, l, 0)/c(G, l, 1) \leqslant 4/3$ (in any network, not necessarily parallel links). Let us denote the price of anarchy by $\rho(G, l) \equiv c(G, l, 0)/c(G, l, 1)$. Then by Roughgarden's result we have that

$$c(G, l, \beta) \leqslant \frac{4}{3+\beta} c(G, l, 1) = \frac{4}{3+\beta} \frac{c(G, l, 0)}{\rho(G, l)}.$$