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Stable allocations of risk ☆

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ABSTRACT

The measurement and the allocation of risk are fundamental problems of portfolio management. Coherent measures of risk provide an axiomatic approach to the former problem. In an environment given by a coherent measure of risk and the various portfolios' realization vectors, risk allocation games aim at solving the second problem: How to distribute the diversification benefits of the various portfolios? Understanding these cooperative games helps us to find stable, efficient, and fair allocations of risk.

We show that the class of risk allocation and totally balanced games coincide, hence a stable allocation of risk is always possible. When the aggregate portfolio is riskless, the class of risk allocation games coincides with the class of exact games. As in exact games any subcoalition may be subject to marginalization even in core allocations, our result further emphasizes the responsibility involved in allocating risk.

1. Introduction

The value of an investment portfolio is subject to all kinds of uncertain events. Firms, banks, or insurance companies (to which we refer by the term portfolios) face *risk* and regulators may require them to hold cash reserves as a cushion against default—this rather unfavorable state of nature—with the precise amounts determined by a *measure of risk*. A measure of risk thereby specifies the minimal amount of cash the regulated agent has to add to his portfolio for his risk to be acceptable to the regulator.

The literature knows of numerous possible ways to measure risk; lately interest shifted to *coherent measures of risk* (Artzner et al., 1999) defined by four axioms: monotonicity, subadditivity, positive homogeneity, and translation invariance. These axioms have been shown to be compatible with a natural general equilibrium approach to measure risk (Csóka et al., 2007b).

Of these axioms, subadditivity expresses that the risk of an aggregate portfolio should not exceed the total risk of the individual subportfolios. In particular, the risk of a firm is less than the sum of the risks of the constituents of the firm. *Risk allocation* then addresses the distribution of the diversification benefits; *risk allocation games* (Denault, 2001) are transferable utility games defined to this purpose.

A risk allocation game assigns to each coalition of portfolios the risk involved in the aggregate portfolio of the coalition. An allocation shows how to share the risk of the aggregate portfolio of the grand coalition among the individual portfolios,

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which has of course consequences on the cash reserves to be held. The allocation makes clear what part of the risk of a firm should be attributed to each of its constituents. A natural question that arises is whether there are stable allocations of risk, allocations of risk that no coalition can object to, that is whether the core of the risk allocation game is non-empty.

We separate the *risk environment* specifying the individual portfolios' realization vectors of discrete random variables and a coherent measure of risk, a real valued function on the realization vectors, from the derived cooperative game that we call *risk allocation game*.

A *totally balanced game* is a cooperative game having a non-empty core in all of its subgames. Totally balanced games arise from a wide range of applications. They coincide with market games (Shapley and Shubik, 1969); also with a special case of market games with a continuum of indivisible commodities: cooperation in fair division (Legut, 1990); they are equivalent to a class of maximum flow problems (Kalai and Zemel, 1982a); and also to permutation games of less than four players (Tijs et al., 1984). Moreover, totally balanced games are generated by linear production games (Owen, 1975), generalized network problems (Kalai and Zemel, 1982b), and controlled mathematical programming problems (Dubey and Shapley, 1984).

We show that the class of risk allocation games coincides with the class of totally balanced games, that is all risk allocation games are totally balanced and all totally balanced games can be generated by a risk allocation game with a properly specified risk environment. This result ensures that a regulator can always allocate risk in a stable way. No matter how the risk environment changes, there is always a core element.

We next provide a linear program such that its optimal objective value can be used to determine whether a given cooperative game is a risk allocation game or not. If the game is a risk allocation game, then an optimal solution to the linear program yields a risk environment that generates the game. We then show how to use the linear program to characterize all risk environments that generate a given totally balanced game.

At last, we focus on games where only the *distribution* of values is uncertain, while the value of the aggregate portfolio is constant over all states of nature. This case is relevant for situations where the risk of the aggregate portfolio is low compared to the risk involved in the individual portfolios. We show that the class of risk allocation games with no aggregate uncertainty coincides with the class of *exact games* (Schmeidler, 1972). As evidenced by the previous paragraphs, there are many applications giving rise to the class of totally balanced games. There are few applications which lead to exact games. The only example we know of is Calleja et al. (2005), who show that the class of multi-issue allocation games coincides with the class of nonnegative exact games.

The fact that each risk allocation game is exact implies that for each coalition there is a core element such that the coalition only gets its stand-alone value. This means that in the case of no aggregate uncertainty, this coalition does not necessarily benefit from the diversification opportunities offered by the aggregate portfolio. As a consequence, the regulator has a high level of discretion in allocating the risk to the individual portfolios.

The structure of the paper is as follows. First we introduce coherent measures of risk, transferable utility games, and risk allocation games. In Section 3 we prove that the class of risk allocation games coincides with the class of totally balanced games and investigate our constructive proof by linear programming. In Section 4 we show that the class of risk allocation games with no aggregate uncertainty coincides with the class of exact games. In Section 5 we conclude.

2. Preliminaries

2.1. Coherent measures of risk

Consider the set $\mathbb{R}^S$ of realization vectors, where S denotes the number of states of nature. State of nature s occurs with probability $p_s > 0$ and $\sum_{s=1}^{S} p_s = 1$. The vector $X \in \mathbb{R}^S$ represents a portfolio's possible profit and loss realizations on a common chosen future time horizon, say at $t = 1$. The amount X_s is the portfolio's payoff in state of nature s. Negative values of X_s correspond to losses. The inequality $Y \geqslant X$ means that $Y_s \geqslant X_s$ for all $s = 1, \dots, S$.

A *measure of risk* is a function $\rho : \mathbb{R}^S \to \mathbb{R}$ measuring the risk of a portfolio from the perspective of the present ($t = 0$). It is the minimal amount of cash the regulated agent has to add to his portfolio, and to invest in a *reference instrument* today, such that it ensures that the risk involved in the portfolio is acceptable to the regulator. We assume that the reference instrument has payoff 1 in each state of nature at $t = 1$, thus its realization vector is $1^S = (1, \dots, 1)^\top$. The reference instrument is riskless in the "classical sense," having no uncertainty in its payoffs. It is most natural to think of it as a zero coupon bond. The price of the reference instrument is denoted by $\delta \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0, \infty)$. We adjust the definition of coherent measures of risk to the discrete case with realization vectors as follows.

Definition 2.1. A function $\rho : \mathbb{R}^S \to \mathbb{R}$ is called a *coherent measure of risk* (Artzner et al., 1999) if it satisfies the following axioms:

1. *Monotonicity*: for all $X, Y \in \mathbb{R}^S$ such that $Y \geqslant X$, we have $\rho(Y) \leqslant \rho(X)$.
2. *Subadditivity*: for all $X, Y \in \mathbb{R}^S$, we have $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$.
3. *Positive homogeneity*: for all $X \in \mathbb{R}^S$ and $h \in \mathbb{R}_+$, we have $\rho(hX) = h\rho(X)$.
4. *Translation invariance*: for all $X \in \mathbb{R}^S$ and $a \in \mathbb{R}$, we have $\rho(X + a1^S) = \rho(X) - \delta a$.