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All equilibria of the multi-unit Vickrey auction ☆

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Abstract

This paper completely characterizes the set of equilibria of the Vickrey auction for multiple identical units when buyers have non-increasing marginal valuations and there are at least three potential buyers. There are two types of equilibria: In the first class of equilibria there are positive bids below the maximum valuation. In this class, above a threshold value all bidders bid truthfully on all units. One of the bidders bids at the threshold for any unit for which his valuation is below the threshold; the other bidders bid zero in this range. In the second class of equilibria there are as many bids at or above the maximum valuation as there are units. The allocation of these bids is arbitrary across bidders. All the remaining bids equal zero. With any positive reserve price equilibrium becomes unique: Bidders bid truthfully on all units for which their valuation exceeds the reserve price.

JEL classification: C72; D44

Keywords: Vickrey auction; Multi-unit auction; Ex-post equilibrium; Reserve price; Uniqueness; Core

1. Introduction

This paper completely characterizes the set of Bayesian Nash equilibria of the Vickrey auction for multiple identical units when buyers have non-increasing marginal valuations and there are at least three potential buyers. Equilibria fall

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into two classes[1]: In the first class, there is positive probability that there are positive bids below the maximum valuation. In this class, there is a threshold for valuations such that all bidders bid truthfully on any unit for which they have a valuation exceeding the threshold. Furthermore, there is a distinct bidder who bids the threshold value on any unit for which his valuation is below the threshold. The remaining bidders bid zero on any unit for which their valuation is below the threshold. In the second class of equilibria, there is zero probability of positive bids below the highest valuation. In this class, each bidder bids at or above the highest valuation on some number of units and bids zero on the remaining units in such a manner that the total number of positive bids across all bidders equals the number of units that are for sale. In any equilibrium, except the conventional equilibrium in dominant strategies, there is positive probability that a bidder wins a unit at a price of zero. In this sense all of these equilibria are collusive.

We also observe that all equilibria of the Vickrey auction are *ex-post* equilibria, i.e. bidders have no incentive to change their behavior even after all private information is revealed and therefore suffer no regret. Indeed, the entire set of equilibria within the first class remain equilibria for *any* change of the distribution function of bidders' valuations, including changes that affect the support of the distribution of bidders' valuations.

With any positive reserve price equilibrium becomes unique: Bidders bid truthfully on all units for which their valuation exceeds the reserve price. From this perspective, our result can be interpreted as providing an alternative foundation for the focus on the truthful-bidding equilibrium. Finally, we establish that only the truthful-bidding equilibrium ensures that the final allocation is in the core for all realizations of valuations.

We are interested in a full characterization of the equilibrium set of the Vickrey auction for the following reasons: First, it is often argued that the single- or multi-unit Vickrey auction is susceptible to collusion. We show that a continuum of collusive equilibria exists, thereby providing some support for this view. (Although as noted above, we also establish that with any effective reserve price and at least three bidders, equilibrium is unique.) Second, for many distributions of valuations, there is a tension between selecting weakly undominated and payoff dominant equilibria in the multi-unit Vickrey auction. The truthful-bidding equilibrium results in comparatively high prices. In contrast, in the second class of equilibria the bidders obtain the units for free. In this class of equilibria, however, the units are allocated inefficiently across bidders, which reduces the bidders' total expected payoffs. Nevertheless, for many distribution functions bidders prefer equilibria in the second class to truthful bidding.[2] Furthermore, for some distributions of valuations, equilibria in the first class with a strictly positive threshold payoff dominate both truthful bidding and equilibria in the second class. Such equilibria improve the allocation relative to equilibria in the second class while still suppressing prices relative to truthful bidding (see Blume and Heidhues, 2001, for an example in the single-unit case).[3] Third, if the auction is repeated (with an arbitrarily small positive probability), the one-shot equilibria can be used to construct collusive equilibria in which no players uses a weakly dominated strategy for *any* arbitrarily small positive discount factor. Fourth, if collusion is the result of a non-binding agreement, then it seems reasonable to assume that such communication has a small effect on the preferences in the subsequent auction. Any arbitrarily small (psychological) cost of breaking one's word and deviating from the informal agreement, however, transforms all collusive equilibria into strict equilibria.[4] Finally, the issue arouses our intellectual curiosity: We think it is interesting to know the implication of the most fundamental solution concept in game theory for the multi-unit Vickrey auction—especially, given the simple structure of the set of Bayesian Nash equilibria.

Vickrey (1961) introduced the second-price sealed-bid auction for both the single- and the multi-object case. With private values, there is a unique equilibrium in undominated strategies: Bidders bid their valuations. Milgrom (1981) notes the existence of other (asymmetric) equilibria in the single-unit case. For two bidders, Plum (1992) describes yet more equilibria in the single-unit case. Blume and Heidhues (2004) characterize all equilibria of the single-unit Vickrey auction with independent private values and three or more bidders. Blume and Heidhues (2001) also cover

[1] In the introduction we ignore inessential variations of behavior on measure zero sets of valuations. These are explicitly taken into account in Section 4.

[2] Proposition 2 in Blume and Heidhues (2008) shows for the single-unit case that the equilibria in class two payoff dominate the truthful equilibrium for the uniform distribution and any distribution that first-order stochastically dominates it.

[3] Collusion via equilibria in the first class may also be more difficult to detect than collusion via equilibria in the second class because prices vary in this class of equilibria, which provides another reason to study these equilibria.

[4] Think of the dinners held by Judge Gary, chairman of U.S. Steel's board of directors, at the beginning of the 20th century that according to him lead to such mutual respect among steel industry leaders that all industry leaders found the obligation to cooperate "more binding [...] than any written or verbal contract." For a brief discussion of this case and, more generally, the role of "gentlemen agreements" in collusive arrangements see Scherer and Ross (1990, pp. 235–236).