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Can heterogeneity of the near-wellbore rock cause extrema of the Darcian fluid inflow rate from the formation (the Polubarinova-Kochina problem revisited)?

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ABSTRACT

Darcian steady 2-D flow to a point sink (vertical well) placed eccentrically with respect to two circles demarcating zones of contrasting permeability is studied by the methods of complex analysis and numerically by MODFLOW package. In the analytical approach, two conjugated Laplace equations for a characteristic flow function are solved by the method of images, i.e. the original sink is mirrored about two circles that generates an infinite system of fictitious sinks and source. The internal circle of the annulus models formation damage (gravel pack) near the well and the ring-shaped zone represents a pristine porous medium. On the external circle the head (pressure) is fixed and on the internal circle streamlines are refracted. The latter is equivalent to continuity of pressure and normal component of specific discharge that is satisfied by the choice of the intensity and loci of fictitious sinks. Flow net and dependence of the well discharge on eccentricity are obtained for different annulus radii and permeability ratios. A non-trivial minimum of the discharge is discovered for the case of the ring domain permeability higher than that of the internal circle. In the numerical solution, a finite difference code is implemented and compared with the analytical results for the two-conductivity zone. Numerical solution is also obtained for an aquifer with a three-conductivity zonation. The case of permeability exponentially varying with one Cartesian coordinate within a circular feeding contour is studied analytically by series expansions of a characteristic function obeying a modified Helmholtz equation with a point singularity located eccentrically inside the feeding contour. The coefficients of the modified Bessel function series are obtained by the Sommerfeld addition theorem. A trivial minimum of the flow rate into a small-radius well signifies the trade-off between permeability variation and short-cutting between the well and feeding contour.

1. Introduction

Oil, gas and groundwater wells, as well as agricultural drains laid in topsoils, are often constructed, completed and exploited by techniques involving a considerable change of porosity and permeability of the adjacent rock or soil. For example, gravel packing, microbial and chemical fouling of the pack with time, penetration of drilling liquids, perforation shooting, cyclostationary wellbore pressure agitation and other stimulation techniques cause severe alterations of the matrix with ensuing adverse or positive effects on the quantity and quality of fluids abstracted through the boreholes (e.g., Civan, 2000).

Recently conducted laboratory experiments (Turekhanov et al., 2007; De Zwart et al., 2008) and field studies (De Zwart, 2007; van Beek et al., 2009) proved that for a long-term well operation in aquifers with common rocks, the pumping-induced hydraulic gradients oriented towards the bore hole are so high that a thin (~ 10 cm) internal cake of fine particles is formed over years (decades). The constituting fine particles are dislodged from the bulk formation and driven by the Darcian flow to the gravel filter where they deposit in a transition zone between the filter and aquifer, owing to the so-called "bridging" and other micromechanical mechanisms (see De Zwart, 2007, Figs. 2.7, 2.8, 2.10, 21.12, 2.13, 5.44). We recall that a drilling mud makes its own low-permeable cake in the rock adjacent to the borehole, although even without this construction-stage induced cake well clogging is allegedly common (De Zwart, 2007).

The engineeringly induced heterogeneity of the matrix in the vicinity of the borehole is usually assumed to be radial with

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respect to the well axis (Polubarinova-Kochina, 1977). The corresponding skin (cake)—whether it is less or more permeable than the bulk rock—alters the flow pattern (pumping rate—drawdown—pressure) and well productivity and longevity (De Zwart, 2007). However, filtration of the mud during drilling, cementation during well completion, acid penetration or polymer injection in EOR are characterized by non-uniform rock permeability in different azimuthal (with respect to the borehole axis) directions (Schlumberger, 1989; Civan, 2000; Fisher et al., 2000). Turekhanov et al. (2007), De Zwart et al. (2008, Personal Communication, 2009) found that even in experiments designed to be radially (spherically) convergent to a sink (well) the formed cakes in a given borehole section are azimuthally non-uniform, i.e. an ideal radial flow does not really occur.

A non-unidirectional variation of the matrix hydraulic properties is schematically depicted in Fig. 1. Here a horizontal cross-section of a vertical well is shown. A circle of radius r (e.g., a “skin-formation” interface) encompasses a zone, the interior of which has permeability κ_1 (hydraulic conductivity k_1). A well of a radius r_w ($r_w \ll r$) is placed eccentrically with respect to this circle that corresponds, for example, to the schemes in Dobrynin (1988, Fig. 17.23). Generally, the shape of this k_1-zone around the well is more complex than an eccentric circle of our Fig. 1 (see, for instance, Dobrynin, 1988, Fig. 17.28 Houben, 2006, or Fisher et al., 2000). To the best of our knowledge, only for circular heterogeneities in Fig. 1 analytical solutions are available.

A circle of radius R serves as a feeding contour along which pressure (reservoir engineering)/hydraulic head (groundwater hydrology), h, is a given constant ($=0$). For oil production wells this circle physically corresponds to an oil-water contact. A well is almost always drilled eccentrically with respect to the feeding contour, which demarcates the “oil island”, usually trapped in an anticline (see, e.g., Dake, 1978). Fluid motion to the well in the pay/aquifer is 3-D and transient but a common simplification is to assume that the well is perfect (penetrates the whole formation) and the feeding contour is static (e.g., Krylov et al., 1948; Muscat, 1949; Polubarinova-Kochina, 1977).

The annulus between the two circles in Fig. 1 has permeability κ_2 (hydraulic conductivity k_2), which is the matrix original value (undamaged and unstimulated). Other interpretations of k_1, k_2-zonation in Fig. 1 are possible (see, e.g., De Zwart, 2007; Ding and Renard, 2005; Rahman et al., 2007). The head on the well contour is $-H_w$ ($H_w > 0$ is a given constant) and the pumping rate is Q (per unit length in the direction perpendicular to the plane of Fig. 1, i.e. we assume that the well is screened throughout the whole depth of an aquifer/pay).

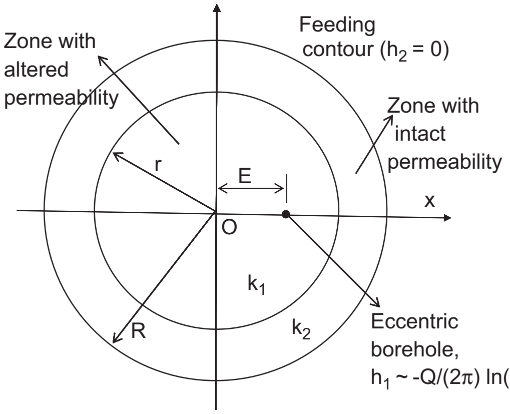

Fig. 1. Plane section of an eccentric well and two adjacent domains of contrasting permeability.

If $E=0$ in Fig. 1 then flow towards the well is trivially radial with straight streamlines and circular isobars (equipotential lines) converging to the sink. Flow to perfect wells as line sinks placed eccentrically with respect to its feeding contour but in homogeneous pays ($k_1=k_2$ in Fig. 1) has been analytically studied by Polubarinova-Kochina (1977). Lu and Tiab (2008) took into account the near-wellbore heterogeneity by a pseudo-skin factor of partially penetrating wells. Chen and Chang (2006) studied flow with well skin thicknesses varying in direction perpendicular to Fig. 1. In none of mathematical models, which we are aware of, a full 2- and 3-D refraction of the incident flow from the feeding contour on the near-wellbore heterogeneities ($k_1 \neq k_2$ in Fig. 1) has been taken into account.

Polubarinova-Kochina (1942) pioneered in analytical assaulting the full refraction problem with genuinely swerving streamlines close to the well. She did not give any derivations to the problem in Fig. 1 and presented final formulae in terms of the specific discharge. In Polubarinova-Kochina books published in Russian (1952) and its translation in English (1962) the formulae are reproduced in a form different from the original Polubarinova-Kochina (1942) paper and also without derivations. Unfortunately, in the latest Polubarinova-Kochina (1977) version of the book there were annoying mistakes in the formulae, which again changed their mathematical form. Neither pressure/head/streamlines nor specific discharge distributions were reported by Polubarinova-Kochina (1942, 1952, 1962, 1977). Anderson (2002) re-derived Polubarinova-Kochina (1962) solution but the derivations were again not reported. He also considered a more general case of two sinks placed symmetrically with respect to the origin of coordinates in Fig. 1 that models a silt cake on the bottom of a river. Anderson (2002) integrated Polubarinova-Kochina (1962) complex velocity field, obtained the complex potential and reconstructed the flow net.

We note that this net is difficult to perceive based on the known hydrogeological “rule of thumb”. We recall that this “rule”—routinely applied to common lamination of commingled aquifers-aquitards in hydrogeology—predicts that in highly/low permeable layers streamlines are quasi-parallel/quasi-orthogonal to the bedding. In this sense an eccentric sink tapping a heterogeneous domain in Fig. 1 is similar to wells placed in fractured reservoirs with impeding gadgets (e.g., elastomers, Kacimov et al., 2009a, b), where the flow net and specific discharge field are not intuitively clear prior to a detailed solution of a boundary-value problem.

It is noteworthy that standard FDM–FEM codes encounter problems in describing the fine features of the flow close to curved interfaces between two different porous zones, where soil/rock particles are most prone to suffosion and other erosion phenomena generated by high gradients of pore pressure. Numerical packages should, in our opinion, be tuned up to adequately describe a heterogeneous vicinity of the well bore as in Fig. 1 that is especially important in EOR planning. Secondary/tertiary recovery techniques with an associated sophistry of 2- and 3-phase transient modeling in the bulk of the pay are often futile, despite the filigree of involved numerical codes, because what occurs in the very vicinity of the well (where the numerical codes routinely posit a radial flow and, for instance, “pseudo-skin factor”)—even without major fractures—is drastically different from real non-radial and “non-pseudo” skin phenomena. So, the ignorance of the near-well flow results in mis-, over-, and under-estimates of the recovery parameters (Kacimov et al., 2009b).

Non-radial 2-D flows in porous media, whose permeability varies in space, have been analytically studied by Alferov and Ryashentsev (1973), Anderson (2000), Bakker and Nieber (2004), Gheorgita (1966), Golubev and Tumashev (1972), Osyatinskii (1969), Polubarinova-Kochina (1977), Warrick and Knight (2003),