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A general approach to full-range tail dependence copulas

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ABSTRACT

Full-range tail dependence copulas have recently been proved very useful for modeling various dependence patterns in the joint distributional tails. However, there are only a few applicable candidate models that have the full-range tail dependence property. In this paper, we present a general approach to constructing bivariate copulas that have full-range tail dependence in both upper and lower tails and are able to account for both reflection symmetry and reflection asymmetry. The general approach is based on mixtures of positive regularly varying random variables, and the full-range tail dependence property is established for such a general model. In order to construct copulas that possess the above dependence properties and are fast to compute, we construct a full-range tail dependence copula based on mixtures of Pareto random variables. We derive dependence properties of the proposed copula, and the extreme value copula based on it. A comparison with the full-range tail dependence copula proposed in Hua (2017) has been conducted, and the computational speed has been largely improved by the copula proposed in the current paper.

1. Introduction

Modern financial and insurance risk management practices often require ones to take multiple dependent risks into account. To better account for the stochastic nature behind dependent risks, one needs to model dependence beyond the multivariate Gaussian framework. This has led to tons of research works dealing with non-Gaussian dependence modeling. However, modeling dependence is not an easy task. Although there is only one way to define bivariate independence, dependence can be formulated in an unlimited number of ways. The notion of copula was first proposed in Sklar (1959), and it has evolved as a widely accepted tool in modern statistical dependence analysis. Among many of the advantages, the capability in modeling a rich variety of tail behavior, such as tail dependence versus tail independence, and reflection symmetry versus reflection asymmetry, is perhaps one of the most desirable capacities that set the copula approach apart from many other multivariate analysis frameworks (Brechmann and Schepsmeier, 2013). The notion of copula is a mathematical object that is convenient for illustrating the dependence structures globally and locally. An n-variate function $C : [0, 1]^n \to [0, 1]$, with $n \in \{2, 3, \dots\}$, is a copula if it is grounded, n-increasing, and has uniform margins (see, Nelsen, 2006).

Along with the increasing complexity of (re)insurance products as well as high competition amongst (re)insurers, more flexible copula models possessing sufficient mathematical tractability and being able to capture a wide range of (tail) dependence patterns are in high demand within the community of risk analysts. To this end, vine copulas and factor copulas have been frequently adopted by academic actuaries to construct multivariate models containing more realistic dependence structures. For both of these two approaches, bivariate copulas act as the building blocks. More specifically, the vine copula approach uses a cascade of bivariate copulas to build up intricate multivariate structures via graphical models, and the factor copula approach achieves different dependence structures by mixing bivariate copulas with common latent factors. We refer the reader to Kurowica and Joe (2011) for a recent review of the former approach, and Krupskii and Joe (2013) for the later one. To implement these commonly-used multivariate copula models, one often needs to select candidate bivariate copulas from many existing parametric copula families, and varying bivariate copulas may be selected for different pairs. A very flexible bivariate copula can itself become a welcome candidate in the pool of bivariate copulas. Moreover, a parsimonious multivariate copula can be constructed simply based on one single bivariate copula family as long as it is sufficiently flexible. With such a parsimonious dependence model, on the one hand there is no need to select the bivariate copulas among many different candidate copulas, while on the other hand the comparison of dependence

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properties among different pairs becomes more straightforward and meaningful, as all pairs are modeled based on the same copula family. In our opinion, the following two desirable criteria at least need to be satisfied in order to obtain a very flexible copula family:

(A1) covering full-range tail dependence in both upper and lower tails (see, Section 2 for more discussions about the notion of full-range tail dependence);
(A2) accounting for both reflection symmetry and reflection asymmetry, meaning that, $C(u_1, u_2)$ and $u_1 + u_2 - 1 + C(1 - u_2, 1 - u_1)$ may or may not equal for any $u_1, u_2 \in [0, 1]$.

The first criterion removes the obstacle that most existing copulas are only able to model either asymptotic dependence or asymptotic independence. We note that a copula with full-range tail dependence in the upper tail can improve the assessment for dependent high-risk scenarios (see, Hua and Xia, 2014). The second criterion is not about the property of full-range tail dependence, but (A2) is very important for real-world applications as it greatly improves the model's flexibility in accounting for both upper and lower tails. Bivariate copulas with the aforementioned properties are useful not only in constructing multivariate dependence models mentioned above, but also in directly accounting for a bivariate dependence structure; bear in mind that there are still many real applications that only involve bivariate dependence and such flexible full-range tail dependence copulas can be helpful in improving the performance for such cases. Practically, the debate that the Gaussian copula lacks sufficient tail dependence to price credit default derivatives (Salmon, 2012) motivates the need of full-range tail dependence copulas. Regarding (A2), financial and insurance data often appears to be asymmetric between upper and lower tails (see, e.g., Okimoto, 2008). Therefore, one needs to choose models that are capable of modeling upper and lower tails in comparable ways.

All in all, copula functions satisfying (A1) and (A2) are highly relevant in actuarial and risk management applications (see, e.g., Hua and Joe, 2011a; Hua and Xia, 2014). But copulas carrying such properties are not common in the literature, especially when we require that such copulas can be implemented for real applications. In this paper, we aim at constructing new bivariate copula families that satisfy the criteria (A1) and (A2). More importantly, we require that the resulting copulas should not only satisfy the above conditions theoretically, but also be computable at a reasonably fast speed so that data analytic tools can be further developed based on such copulas; it turns out that the latter is often more challenging to achieve.

Recently, Hua (2017) proposed a bivariate copula that satisfies (A1) and (A2) at the same time, and moreover the copula is computable and feasible for real applications. The copula developed in Hua (2017) is constructed by using the mixtures of two gamma and two exponential random variables (rv's), and it is thus referred to as the GGEE copula in what follows. The GGEE copula was shown to satisfy (A1) and (A2), and be superior in modeling dependent financial and insurance data than some other commonly-used copulas (Hua, 2017). For the GGEE copula, an unideal aspect that surely can be outweighed by its remarkable performance is the computational speed, which is acceptable but may not be fast enough for complex applications of modeling high-dimensional dependence structures. Specifically, the hypergeometric function and the Appell's F_1 function involved in the representation of the GGEE copula slow down the computational speed.

The key mechanism hid under the GGEE copula to induce full-range tail dependence is that, two bivariate random vectors with regularly varying tails are mixed. One of the random vectors is comonotonic (Dhaene et al., 2002), and the other is independent. The interplay between the comonotonic and the independent random pairs manipulates the extremal co-movements of the underlying copula, depending on which one dominates the tails. Inspired by the aforementioned observations, our objective in this paper is to establish a general approach to constructing a tractable class of bivariate copulas that possess full-range tail dependence. As a byproduct, we shall identify a subclass of such copulas, and this subclass should also possess the feature of reflection symmetry and asymmetry; this is again motivated by the model in Hua (2017).

Moreover, the general approach proposed in this paper leads to a new copula model induced by a Pareto mixture model. To facilitate further discussions and make our notation consistent with the one used in Hua (2017), we will refer the copula to be proposed in this paper as the PPPP copula, since it is induced by a random vector whose each margin is a mixture of four independent Pareto rv's. The PPPP copula is surprisingly tractable, and many of its distributional properties have closed-form expressions. These attractive properties further attribute to the fast computational speed of the PPPP copula, and make the copula potentially useful for more complex applications.

We openly admit that, in the same vein as the Gaussian copula and the Student-t copula which are used broadly in banking and insurance, the copulas implied by the proposed structure may not have simple expressions that only involve basic operations. This is however a result of the fact that the proposed model is constructed by using mixtures of regularly varying rv's, and it is generally difficult to obtain explicit formulas for the univariate marginal inverse cumulative distribution functions (cdf's) and thus the corresponding copulas. To derive the mathematical properties of these copulas, instead of directly working on the copula functions of interest, we usually need to start with the stochastic representation behind them. In order to apply the maximum likelihood method to calibrate these copulas, one needs to seek appropriate distribution candidates so that the copula models can be fast computed. This is of enormous importance for real-world applications and thus one of the core objectives in this current paper. For the PPPP copula, our implementation has shown that the marginal inverse cdf's can be easily solved by employing commonly used numerical methods. It is noteworthy that the PPPP copula provides a flexible tool for modeling dependence structures between various univariate margins, while the underlying scale mixture model gives rise to a meaningful interpretation of the copula. Therefore, with the goal of constructing superior dependence analysis tool in mind, we focus on the copula itself and derive a number of copula-based properties such as the parameters for quantifying tail dependence, Kendall's τ, Spearman's ρ, and the corresponding extreme value copulas.

Finally, let us document herein some other potential methods of constructing full-range tail dependence copulas. Besides the mixture model that we are going to propose in this current paper, full-range tail dependence copulas might be constructed by using asymmetrizations (Liebscher, 2008), multiple factor models (Su and Furman, 2017), patchwork models (Durante et al., 2013), some generalized Archimedean copulas (Hofert and Vrins, 2013), etc. We emphasize again that, compared to the aforementioned methods, the proposed approach helps us develop a computable class of parametric models so that a wide range of tail dependence in both lower and upper tails and tail asymmetry/symmetry can both be properly covered.

The paper is then organized as follows. After setting up some of the basic notation in Section 2, we present the general approach to constructing full-range tail dependence copulas in Section 3. In Section 4, we propose and study the PPPP copula in details. Specifically, various distributional properties are proved for the PPPP copula, and the corresponding copula domain of attraction is investigated. A numerical study of actuarial interest is conducted to demonstrate the practical usefulness of the PPPP copula. Section 5 concludes the paper with further discussions. A few extra comments and results are contained in Appendix A, and tedious proofs are relegated to Appendix B in order to facilitate the reading.