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Purchasing casualty insurance to avoid lifetime ruin

Virginia R. Young

Department of Mathematics, University of Michigan, Ann Arbor, Michigan, 48109, USA

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ABSTRACT

We determine the optimal strategies for purchasing deductible insurance and for investing in a risky financial market in order to minimize the probability of lifetime ruin when an individual is subject to an insurable loss that occurs at a Poisson rate. We specialize to the case for which the casualty loss is constant and insurance is priced actuarially fairly. We learn that the optimal deductible strategy is for the individual to purchase no insurance when her wealth is below a so-called *buy level*. However, when wealth is greater than the buy level, the individual optimally purchases full insurance coverage.

1. Introduction

We determine the optimal strategies for purchasing deductible insurance and for investing in a risky financial market in order to minimize the probability of lifetime ruin when an individual is subject to an insurable loss that occurs at a Poisson rate. We specialize to the case for which the insurable loss is constant and insurance is priced actuarially fairly. We learn that the optimal deductible strategy is for the individual to purchase no insurance when her wealth is below a so-called *buy level*. However, when wealth is greater than the buy level, the individual optimally purchases full insurance coverage.

The work in this paper combines three areas of research. One area is classical ruin theory, although most work in that area is from the viewpoint of an insurance company facing possible ruin. As in classical ruin theory, we assume that the individual's wealth is subject to a loss that follows a compound Poisson process. As in more recent work in ruin theory, we allow the agent (an individual in our case) to invest in a risky financial market and to buy insurance to mitigate the loss. However, because we take the viewpoint of an individual, the game ends if the individual dies before ruin, a feature not seen in classical ruin theory. That said, the Gerber–Shiu function (Gerber and Shiu, 1998) includes a discount rate when considering the time of ruin, which is mathematically equivalent to a constant force of mortality in our setting.

The second area is optimally controlling wealth to reach a goal or to avoid ruin. Research on this topic began with the seminal work of Dubins and Savage (1965, 1976) and continued with the work of Pestien and Sudderth (1985), Orey et al. (1987), Sudderth and Weerasinghe (1989), Kulldorff (1993), Karatzas (1997), and Browne (1997, 1999a, b). Milevsky et al. (1997) and Milevsky and Robinson (2000) introduced the notion of *lifetime ruin*, namely, the event that an individual ruins before she dies, and Young (2004) used that concept to determine the optimal investment strategy in a risky financial market to minimize the probability of lifetime ruin.

The third area is determining insurance that optimizes a given criterion. We extend the work of Young (2004) by including an insurable loss that occurs at a Poisson rate along with casualty insurance to mitigate the loss. The individual chooses the per-loss deductible at each moment of time and pays for that deductible insurance via a continuous premium rate. This paper is closely aligned with that of Moore and Young (2006), in which they found the optimal per-loss indemnity contract for the compound Poisson risk process, but with a more general severity distribution and with a positive proportional risk loading, to maximize expected utility of lifetime consumption and bequest.

To the best of our knowledge, this paper is the first to consider optimal insurance to minimize the probability of *lifetime* ruin. Most work in optimal insurance takes one of two forms, namely, either finding the optimal indemnity contract in a one-period model to optimize some criterion subject to a premium functional, as in

E-mail address: vryoung@umich.edu.

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Promislow and Young (2005b), or finding the optimal dynamic reinsurance (often limited to proportional reinsurance) to optimize some criterion, as in Promislow and Young (2005a).

The rest of the paper is organized as follows. In Section 2, we present the financial and insurance market in which the individual invests and purchases insurance, we formalize the problem of minimizing the probability of lifetime ruin, and we give a verification lemma that will help us to find that minimum probability, along with the optimal strategies for investing in the financial market and for purchasing per-loss deductible insurance. In Section 3, we solve the problem of minimizing the probability of lifetime ruin when the rate of consumption (or income) is zero; we separate this case because we can solve it explicitly. Sections 4 and 5 parallel Section 3 for a positive rate of income and for a positive rate of consumption, respectively. In those cases, we solve the problem by solving for the Legendre concave dual of the minimum probability of lifetime ruin.

In Section 6, we examine properties of the solution obtained in Sections 3 through 5. Section 7 concludes the paper.

2. Statement of the problem and verification lemma

In this section, we define the financial and insurance markets in which the individual invests and purchases insurance. Then, we state the optimization problem the individual faces and present a verification lemma we use to solve the optimization problem.

2.1. Financial and insurance markets and the probability of lifetime ruin

We assume the individual invests her wealth in a risky financial market, and her wealth is also subject to an insurable loss occurring according to a compound Poisson process. She consumes at the constant rate c; if c is negative, then we say the individual has net income $\gamma = -c > 0$. In Sections 3, 4, and 5, we separately consider $c = 0$, $c < 0$, and $c > 0$, respectively.

The individual invests in a Black–Scholes financial market with one riskless asset earning interest at the rate $r \geq 0$ and one risky asset whose price process $\{S_t\}_{t \geq 0}$ follows geometric Brownian motion:

$$dS_t = \mu \, S_t \, dt + \sigma \, S_t \, dB_t,$$

in which $\{B_t\}_{t \geq 0}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$, with $\mu > r$ and $\sigma > 0$. Let π_t denote the dollar amount invested in the risky asset at time $t \geq 0$. An investment policy $\Pi = \{\pi_t\}_{t \geq 0}$ is *admissible* if it is an $\mathbf{F}$-progressively measurable process satisfying $\int_0^t \pi_s^2 \, ds < \infty$, almost surely, for all $t \geq 0$.

The insurable loss follows a compound Poisson process, also living on the filtered probability space. The cumulative loss at time $t \geq 0$ is given by

$$\sum_{i=1}^{N_t} Y_i,$$

in which $Y_1, Y_2, \ldots$ are iid positive loss random variables, and they are independent of the Poisson process $\{N_t\}_{t \geq 0}$, as well as the Brownian motion driving the risky asset's price process. The Poisson rate is denoted by $h > 0$, h for *hazard*.

Assumption 2.1. For simplicity, in this paper, we assume that $Y_i \equiv \ell$ for all $i = 1, 2, \ldots$. Even under this simplifying assumption, the problem's solution is non-trivial and (somewhat) difficult to obtain and analyze. □

The individual can purchase deductible insurance with deductible d_t at time $t \geq 0$;[1] thus, the individual can change her deductible instantaneously.[2] Let W_t denote the wealth of the individual at time $t \geq 0$. A deductible strategy $D = \{d_t\}_{t \geq 0}$ is *admissible* if it is an $\mathbf{F}$-progressively measurable process satisfying $d_t \in [0, \ell \wedge W_t]$ or $d_t = \ell$, almost surely, for all $t \geq 0$.

Remark 2.1. An individual who values more wealth over less, such as a ruin-probability minimizer, will never buy deductible insurance with deductible *greater* than existing wealth because if she were to have a loss with $d_t > W_t$, she would surely ruin; thus, she would have been better off not buying any insurance. Also, we assume that $d_t \leq \ell$; otherwise, the individual would continually gain from the negative premium rate. Finally, we assume that $d_t \geq 0$ to prevent moral hazard; specifically, if $d_t < 0$, then the individual would gain net wealth of $-d_t$ if a loss were to occur. □

In exchange for insurance with deductible d, the individual pays premium at the actuarially fair rate of $h(\ell - d)$. Thus, under deductible insurance, wealth follows the dynamics

$$\begin{aligned} dW_t &= (rW_t + (\mu - r)\pi_t - c - h(\ell - d_t))\, dt \\ &\quad + \sigma \, \pi_t \, dB_t - (\ell \wedge d_t) dN_t. \end{aligned} \tag{2.1}$$

Denote the future lifetime random variable of the individual by τ_d, which is independent of the Brownian motion and the compound Poisson loss process. We assume τ_d follows an exponential distribution with mean $1/\lambda$. The individual seeks to minimize the probability that her wealth becomes negative before she dies. If we define $\tau_0 = \inf\{t \geq 0 : W_t < 0\}$, then she minimizes the probability that $\tau_0 < \tau_d$; thus, the value function is given by

$$\psi(w) = \inf_{(\Pi, D)} \mathbf{P}^w \left(\tau_0 < \tau_d \right), \tag{2.2}$$

in which $\mathbf{P}^w$ denotes conditional probability given $W_0 = w \geq 0$.

Remark 2.2. If wealth is large enough, say at least w_s (subscript s for *safe*), then the individual can invest all her wealth in the riskless asset with the interest income sufficient to cover her consumption and insurance premium with deductible $d = 0$. That is, wealth w_s generates interest of $r w_s = c + h\ell$, or

$$w_s = \frac{c + h\ell}{r}, \tag{2.3}$$

which we call the *safe level*. Thus, $\psi(w) = 0$ if $w \geq w_s$, and by definition, $\psi(w) = 1$ if $w < 0$. It remains for us to determine $\psi(w)$ for $0 \leq w < w_s$.

In Section 4, we consider the case for which $c < 0$, that is, the individual has net positive income. In that case, we assume that $\gamma = -c < h\ell$; otherwise, net income would be sufficient to cover the full cost of insurance, and ruin would be impossible. Note that $w_s \leq 0$ if $-c \geq h\ell$. □

Remark 2.3. By reasoning probabilistically, we deduce that the minimum probability of lifetime ruin ψ increases as the Poisson rate of casualty loss h increases because losses become more frequent, insurance becomes more expensive, and the safe level increases. Similarly, we deduce that ψ decreases as the mortality rate λ increases because the individual becomes more likely to die before ruin. □

[1] Note that deductible $d = 0$ corresponds to full insurance; $d = \ell$, to no insurance.

[2] In other work, we consider a discrete-time version of the lifetime ruin problem in which the individual buys casualty insurance at the beginning of each period to be in effect for the entire period.