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# Valuation of variable annuities with Guaranteed Minimum Withdrawal Benefit under stochastic interest rate

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ABSTRACT

This paper develops an efficient direct integration method for pricing of the variable annuity (VA) with guarantees in the case of stochastic interest rate. In particular, we focus on pricing VA with Guaranteed Minimum Withdrawal Benefit (GMWB) that promises to return the entire initial investment through withdrawals and the remaining account balance at maturity. Under the optimal (dynamic) withdrawal strategy of a policyholder, GMWB pricing becomes an optimal stochastic control problem that can be solved using backward recursion Bellman equation. Optimal decision becomes a function of not only the underlying asset but also interest rate. Presently our method is applied to the Vasicek interest rate model, but it is applicable to any model when transition density of the underlying asset and interest rate is known in closed-form or can be evaluated efficiently. Using bond price as a numéraire the required expectations in the backward recursion are reduced to two-dimensional integrals calculated through a high order Gauss–Hermite quadrature applied on a two-dimensional cubic spline interpolation. The quadrature is applied after a rotational transformation to the variables corresponding to the principal axes of the bivariate transition density, which empirically was observed to be more accurate than the use of Cholesky transformation. Numerical comparison demonstrates that the new algorithm is significantly faster than the partial differential equation or Monte Carlo methods. For pricing of GMWB with dynamic withdrawal strategy, we found that for positive correlation between the underlying asset and interest rate, the GMWB price under the stochastic interest rate is significantly higher compared to the case of deterministic interest rate, while for negative correlation the difference is less but still significant. In the case of GMWB with predefined (static) withdrawal strategy, for negative correlation, the difference in prices between stochastic and deterministic interest rate cases is not material while for positive correlation the difference is still significant. The algorithm can be easily adapted to solve similar stochastic control problems with two stochastic variables possibly affected by control. Application to numerical pricing of Asian, barrier and other financial derivatives with a single risky asset under stochastic interest rate is also straightforward.


## 1. Introduction

The world population is getting older fast with life expectancy raising to above 90 years in some countries. Longevity risk (the risk of outliving one's savings) became critical for retirees. Variable annuity (VA) with living and death benefit guarantees is one of the products that can help to manage this risk. It takes advantage of market growth and at the same time provides protection of the savings. VA guarantees are typically classified as guaranteed minimum withdrawal benefit (GMWB), guaranteed minimum accumulation benefit (GMAB), guaranteed minimum income benefit (GMIB), and guaranteed minimum death benefit (GMDB). A good overview of VA products and the development of their market can be found in Bauer et al. (2008), Ledlie et al. (2008) and Kalberer and Ravindran (2009). Insurers started to sell these types of products from the 1990s in United States. Later, these products became popular in Europe, UK and Japan. The market of VAs is very large, for example, sales of these contracts in United States between 2011 and 2013 averaged about $160 billion per year according to the LIMRA (Life Insurance and Market Research Association) fact sheets.

For clarity and simplicity of presentation, in this paper we consider a VA contract with a very basic GMWB guarantee that promises to return the entire initial investment through cash withdrawals during the policy life plus the remaining account balance at maturity, regardless of the portfolio performance. Thus even when the account of the policyholder falls to zero before maturity, GMWB feature will continue to provide the guaranteed cashflows.

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GMWB allows the policyholder to withdraw funds below or at the contractual rate without penalty and above the contractual rate with some penalty. If the policyholder behaves passively and makes withdrawals at the contractual rate defined at the beginning of the contract, then the behavior of the policyholder is called ***static***. In this case the paths of the wealth account can be simulated and a standard Monte Carlo (MC) simulation method can be used for GMWB pricing. On the other hand if the policyholder optimally decides the amount to withdraw at each withdrawal date, then the behavior of the policyholder is called ***dynamic***. Under the optimal withdrawal strategy, the pricing of variable annuities with GMWB becomes an optimal stochastic control problem. This problem cannot be solved by a standard simulation-based method such as the well known Least-Squares MC method introduced in Longstaff and Schwartz (2001). This is because the paths of the underlying wealth process are altered by the optimal cash withdrawals that should be found from the backward in time solution and the underlying wealth process cannot be simulated forward in time. However, it should be possible to apply control randomization methods extending Least-Square MC to handle optimal stochastic control problems with controlled Markov processes recently developed in Kharroubi et al. (2014); though the accuracy and robustness of this method for GMWB pricing has not been studied yet.

It is important to note that the fair fee for the VA guarantee obtained under the assumption that the policyholders behave optimally to maximize the value of the guarantee is an important benchmark because it is a worst case scenario for the contract writer. That is, under the no-arbitrage assumption, if the guarantee is perfectly hedged then the issuer will receive a guaranteed profit if the policyholder deviates from the optimal strategy. Pricing under any other strategy will lead to smaller fair fee. Of course, the strategy optimal in this sense may not be optimal to the policyholder under his circumstances and preferences. On the other hand, secondary markets for equity linked insurance products are growing and financial third parties can potentially generate guaranteed profit through hedging strategies from VA guarantees which are not priced according to the worst case assumption about the optimal strategies. There are a number of studies considering these aspects and we refer the reader to Shevchenko and Luo (2016) for discussion of this topic and references therein.

Pricing of VA with a GMWB feature assuming constant interest rate has been considered in many papers over the last decade. For example, Milevsky and Salisbury (2006) developed a variety of methods for pricing GMWB products. In their *static* withdrawal approach the GMWB product is decomposed into a Quanto Asian put option plus a generic term-certain annuity. They also considered pricing when the policyholder can terminate (surrender) the contract at the optimal time, which leads to an optimal stopping problem akin to pricing an American put option. Bauer et al. (2008) presents valuation of variable annuities with multiple guarantees via a multidimensional discretization approach in which the Black–Scholes partial differential equation (PDE) is transformed to a one-dimensional heat equation and a quasi-analytic solution is obtained through a simple piecewise summation with a linear interpolation on a mesh. Dai et al. (2008) developed an efficient finite difference algorithm using the penalty approximation to solve the singular stochastic control problem for a continuous time withdrawal model under the optimal withdrawal strategy and also finite difference algorithm for discrete time withdrawal. Their results show that the GMWB values from the discrete time model converge fast to those of the continuous time model. Huang and Forsyth (2012) did a rigorous convergence study of this penalty method for GMWB, and Huang and Kwok (2014) deduce various asymptotes for the free boundaries that separate different withdrawal regions in the domain of the GMWB pricing model. Chen and Forsyth (2008) present an impulse stochastic control formulation for pricing variable annuities with GMWB under the optimal policyholder behavior, and develop a numerical scheme for solving the Hamilton–Jacobi–Bellman variational inequality for the continuous withdrawal model as well as for pricing the discrete withdrawal contracts.

More recently, Azimzadeh and Forsyth (2014) prove the existence of an optimal bang–bang control for a Guaranteed Lifelong Withdrawal Benefits (GLWB) contract. In particular, they find that the holder of a GLWB can maximize the contract writer's losses by only performing non-withdrawal, withdrawal at exactly the contract rate or full surrender. This dramatically reduces the optimal strategy space. However, they also demonstrate that the related GMWB contract is not convexity preserving, and hence does not satisfy the bang–bang principle other than in certain degenerate cases. GMWB pricing under bang–bang strategy was studied in Luo and Shevchenko (2015c), and Huang and Kwok (2015) have developed a regression-based MC method for pricing GLWB. For GMWB under the optimal withdrawal strategy, the numerical evaluations have been developed by Dai et al. (2008) and Chen and Forsyth (2008) using finite difference PDE methods and by Luo and Shevchenko (2015a) using direct integration method. Pricing of VAs with both GMWB and death benefit (both under static and dynamic regimes) has been developed in Luo and Shevchenko (2015b).

Some withdrawals from the VA type contracts can also attract country specific government additional tax and penalty. Recently, Moenig and Bauer (2015) demonstrated that including taxes significantly affects the value of withdrawal guarantees in variable annuities producing results in line with empirical market prices. These matters are not considered in our paper but can be handled by the numerical methodology developed here.

In the literature on pricing GWMB, interest rate is typically assumed to be constant. Few papers considered the case of stochastic interest rate. In particular, Peng et al. (2012) considered pricing GMWB under the Vasicek stochastic interest rate in the case of *static* withdrawal strategy; they derived the lower and upper bounds for the price because closed-form solution is not available due to withdrawals from the underlying wealth account during its stochastic evolution. Bacinello et al. (2011) considered stochastic interest rate and stochastic volatility models under the Cox–Ingersoll–Ross (CIR) models. They developed pricing in the case of static policyholder behavior via the ordinary MC method and *mixed* valuation (where the policyholder is *semiactive* and can decide to surrender the contract at any time before the maturity) is performed by the Least-Squares MC. Forsyth and Vetzal (2014) considered modeling stochasticity in the interest rate and volatility via the Markov regime switching models and developed pricing under the static and dynamic withdrawal strategies. Under this approach, the interest rate and volatility are assumed to have the finite number of possible values and their evolution in time is driven by the finite state Markov chain variable representing possible regimes of the economy.

In this paper, we develop direct integration method for pricing of VAs with guarantees under the *dynamic* and *static* withdrawal strategies when the interest rate follows the *Vasicek stochastic interest rate* model. In the case of general stochastic processes for the underlying asset and interest rate, numerical pricing can be accomplished by PDE methods that become slow and difficult to implement in the case of two and more underlying stochastic variables. Our method is developed for the case when the bivariate transition density of the underlying asset and interest rate are known in closed-form or can be evaluated efficiently. That is, it should be possible to apply this method to the case of, for example, CIR stochastic interest rate model. Using change of numéraire technique with bond price as a numéraire, the required