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Robust Bayesian estimation and prediction of reserves in exponential model with quadratic variance function

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ABSTRACT

The exponential families with quadratic variance function, conjugate families of priors and square loss function is applied to the prediction of claim reserves. The robustness with respect to the priors is considered. The uncertainty of the prior information is modeled by two different classes of priors. The posterior regret Γ-minimax estimators and predictors are constructed.

1. Introduction

The prediction of claim reserves is one of the most important problems in the insurance mathematics and is the main task of insurance actuaries. There are a lot of models, methods and algorithms setting claim reserves. A wide range of stochastic methods is presented in England and Verrall (2002) and Wüthrich and Merz (2008). Many of them are based on individual loss development ratios which have been observed so far, see chain ladder (Mack, 1993) and Bornhuetter–Ferguson methods (Bornhuetter and Ferguson, 1972), lognormal model (Han and Gau, 2008) and credibility models (Gisler and Wüthrich, 2008). The main aim is to predict an appropriate random variable which describes future payments or estimate parameters of that random variable distribution.

In this paper we deal with Bayesian models which can be applied to the prediction of claim reserves. We consider the exponential families with quadratic variance function (EQVF) with associated conjugate families of priors (for definitions see Section 2) and the square loss function. We generalize the family EQVF (known in literature) adding one more parameter r. Thanks to this we obtain model of reserves satisfying conditions about expected value and variance (see Eqs. (4) and (5)) like in the classical chain ladder model. But prior distribution provides a mechanism for incorporating information from previous studies. Loss reserving analysts can have experience and background knowledge and they can apply it by a prior distribution. The model can be applied to the prediction of number of claims (for example applying Poisson or negative binomial distribution) or amount of reserve (for example normal or gamma distribution). In this model the Bayesian estimator of the chain ladder factor and predictor of reserves can be written as credibility formulas. The presented model is connected with exponential dispersion families but in our model we consider the prior distribution exactly of chain ladder factor. Thus it has clear interpretation and its elicitation may be easier. It is some generalization of Bayesian models presented in Wüthrich and Merz (2008), Gisler (2006) and England et al. (2012). Bayesian methods are not new in the problem of prediction of loss reserving. They are considered in Verrall (1990), de Alba (2002), Ntzoufras and Dellaportas (2002), Wuthrich (2007), Meyers (2009), Peters et al.

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(2009), Merz and Wüthrich (2010), Sánchez and Vilar (2011), Shi et al. (2012), Zhang et al. (2012) and Dong and Chan (2013) among others.

Our aim is not only to present the Bayes estimators and predictors but we also consider the robustness with respect to the priors and construct optimal procedures. In Bayesian statistical inference the arbitrariness of a unique prior distribution is a permanent problem. The uncertainty of the prior information is expressed by using a class Γ of priors. There are several concepts of optimal rules: Γ-minimax rules (e.g. Berger, 1994), conditional Γ minimax rules (Betrò and Ruggeri, 1992; Męczarski, 1993; Boratyńska, 2002b), stable and posterior regret Γ-minimax rules (Zen and DasGupta, 1993; Boratyńska, 1997, 2002a; Kamińska and Porosiński, 2009; Kiapour and Nematollahi, 2011). The general references on robust Bayesian methods are Berger (1994), Rios Insua and Ruggeri (2000) and Boratyńska (2008b). In insurance models the robustness of Bayesian inference is considered in Gómez-Déniz et al. (2002, 2006), Gómez-Déniz (2008), Chan et al. (2008), Boratyńska (2008a) and Karimnezhad and Parsian (2014) among others. There are not many papers dedicated to the problem of reserves prediction where the robustness with respect to priors is considered and the posterior regret Γ-minimax estimators or predictors have not been presented in insurance reserve models so far.

In this paper we consider two different classes of priors and the concept of posterior regret Γ-minimax estimators of the chain ladder factors and predictors of reserves. Their values depend only on the bounds of a set of Bayes actions calculated with respect to the priors belonging to the class Γ. Thus computing a posterior regret Γ-minimax estimator is simple provided that we have procedures to compute the range of posterior expectations. The situation, where there exists a prior in the class Γ such that the posterior regret Γ-minimax estimator or predictor becomes Bayes with respect to this prior, is presented.

2. Exponential family with quadratic variance function

Let a, b, c and $r > 0$ be fixed real numbers such that the set

$$\Theta^{+} = \{\theta \in \mathcal{R} :\ a\theta^2 + b\theta + c > 0\}$$

is non-empty. Let h and q be continuously differentiable real valued functions satisfying the differential equations

$$\frac{h'(\theta)}{h(\theta)} = \frac{-\theta}{a\theta^2 + b\theta + c} \tag{1}$$

and

$$q'(\theta) = \frac{1}{a\theta^2 + b\theta + c}. \tag{2}$$

Let Θ be a maximal interval (θ_0, θ_1) such that $\Theta \subset \{\theta \in \Theta^{+} :\ h(\theta) > 0\}$. Let $\{P_\theta : \theta \in \Theta\}$ be a one-parameter exponential family of probability measures on $\mathcal{R}$ with finite two first moments and with densities of the form

$$f(x|\theta) = g(x)h^r(\theta)e^{q(\theta)x}, \quad x \in \mathcal{R}$$

with respect to some σ-finite measure on $\mathcal{R}$. Then

$$E(X|\theta) = r\theta \quad \text{and} \quad Var(X|\theta) = r(a\theta^2 + b\theta + c).$$

Hence the expected value is a linear and the variance is a quadratic function of a parameter θ and we will say that a random variable X has exponential quadratic variance function distribution with parameters a, b, c, r, θ and denote $X \sim EQVF(a, b, c, r, \theta)$. That family with $r = 1$ has been defined by Morris (1982), Chen et al. (1991) and Eichenauer-Hermann (1991). We made some generalization by adding $r > 0$. Poisson, Gamma, negative binomial distributions are examples of distributions belonging to that exponential family of distributions. For more details see Table 1 at the end of the paper. For detailed characteristics of the family and proofs of properties see Morris (1982) and Chen et al. (1991).

Table 1
Examples of *EQVF* families.

Distribution	a, b, c	$\pi_{\alpha,\beta} \propto$
Normal $N(r\theta, r), \theta \in \mathcal{R}$	$a = b = 0, c = 1$	$e^{-0.5\alpha\theta^2+\beta\theta}$
Poisson $P(r\theta)$ $\theta > 0$	$a = c = 0, b = 1$	$\theta^\beta e^{-\alpha\theta}$
Negative binomial $bin^{-}(r, \frac{1}{\theta+1}), \theta > 0$	$a = b = 1, c = 0$	$(\frac{1}{\theta+1})^\alpha(1-\frac{1}{\theta+1})^\beta$
$Gamma(\frac{r}{a}, \frac{1}{a\theta})$ $\theta > 0$	$a > 0, b = c = 0$	$(\frac{1}{\theta})^{\frac{\alpha}{a}} e^{-\frac{\beta}{a\theta}}$

Let $\Pi(\alpha, \beta)$ be a probability measure on the interval Θ with the Lebesgue density

$$\pi_{\alpha,\beta}(\theta) \propto h^\alpha(\theta)e^{\beta q(\theta)}, \quad \theta \in \Theta,$$

where α, β are real parameters satisfying

$$\alpha > 3a \quad \text{and} \quad \beta > \theta_0(\alpha - 2a) - b \quad \text{and} \\ \beta < \theta_1(\alpha - 2a) - b \tag{3}$$

(see Chen et al., 1991). Then $\Pi(\alpha, \beta)$ is a conjugate prior and given a random sample $\underline{x} = (x_1, x_2, \ldots, x_n)$ from a distribution P_θ the posterior distribution is equal to $\Pi(\alpha+nr, \beta+\sum_{i=1}^n x_i)$ (parameters of the posterior distribution satisfy conditions (3), the proof is similar to Chen et al., 1991) and the Bayes estimator of θ under the square error loss is given by the formula

$$\hat{\theta}^B_\Pi(\underline{x}) = \frac{\beta + b + \sum_{i=1}^n x_i}{\alpha - 2a + nr}.$$

The posterior risk of the estimator is

$$R_{\underline{x}}(\Pi, \hat{\theta}^B_\Pi) = \frac{\left(\beta + b + \sum_{i=1}^n x_i\right)^2 a + b\left(\beta + b + \sum_{i=1}^n x_i\right)(\alpha + nr - 2a)}{(\alpha + nr - 2a)^2(\alpha + nr - 3a)} + \frac{c}{\alpha + nr - 3a},$$

where the posterior risk of an estimator $\hat{\theta}$ under the prior Π and the value x of an observed random variable X is equal to the expected value of square error if θ has the posterior distribution, hence $R_x(\Pi, \hat{\theta}) = E_\Pi((\theta - \hat{\theta})^2|x)$.

3. Posterior regret Γ-minimax estimation and prediction

Now suppose that the prior distribution is not exactly specified and considered the class Γ of priors. We are interested in calculating the posterior regret Γ-minimax (PRGM) decision rule (in our problem estimator or predictor) i.e. the decision rule d^{PR}_Γ satisfying for every value x of an observed random variable X

$$\sup_{\Pi\in\Gamma} r_x(\Pi, d^{PR}_\Gamma(x)) = \inf_{a\in\mathcal{R}} \sup_{\Pi\in\Gamma} r_x(\Pi, a),$$

where $r_x(\Pi, a)$ is a posterior regret equal to

$$r_x(\Pi, a) = R_x(\Pi, a) - R_x(\Pi, d^B_\Pi(x)),$$

$R_x(\Pi, a)$ is a posterior risk of an action (value of an estimator or predictor when the observed random variable X is equal to x) a, and d^B_Π is the Bayes rule. The posterior regret measures the loss of optimality due to choosing a instead of the optimal Bayes action. Given the imprecision in elicitation of a prior, we try to make a decision, and this decision cannot be a Bayes action for every prior in the class Γ. Thus we choose an action (in our problem an estimate or a predictor) which minimizes the maximum loss of optimality in the class Γ, and the largest possible increase in