Insurance: Mathematics and Economics 76 (2017) 172–184

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Mean–variance target-based optimisation for defined contribution pension schemes in a stochastic framework

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A R T I C L E I N F O

Article history:
Received August 2016
Received in revised form August 2017
Accepted 7 August 2017
Available online 18 August 2017

JEL classification:
C61
D81
D90
G11
G22

Keywords:
Mean–variance approach
Defined contribution pension scheme
Stochastic optimal control
Martingale method
Efficient frontier
Ruin probability

A B S T R A C T

We solve a mean–variance optimisation problem in the accumulation phase of a defined contribution pension scheme. In a general multi-asset financial market with stochastic investment opportunities and stochastic contributions, we provide the general forms for the efficient frontier, the optimal investment strategy, and the ruin probability. We show that the mean–variance approach is equivalent to a "user-friendly" target-based optimisation problem which minimises a quadratic loss function, and provide implementation guidelines for the selection of the target. We show that the ruin probability can be kept under control through the choice of the target level. We find closed-form solutions for the special case of stochastic interest rate following the Vasiček (1977) dynamics, contributions following a geometric Brownian motion, and market consisting of cash, one bond and one stock. Numerical applications report the behaviour over time of optimal strategies and non-negative constrained strategies.

1. Introduction and motivation

Defined contribution (DC) pension schemes are becoming more and more important in the pension systems of most industrialised countries and are replacing the defined benefit (DB) schemes that were more frequent in the past. It is well known that the investment risk, which is borne by the sponsor in DB pension schemes, is faced by the member in DC pension schemes and its analysis is therefore of the utmost importance nowadays.

The optimal investment strategy in the accumulation phase (i.e. prior to retirement) in a DC framework has been derived in the literature with a variety of objective functions (mainly maximisation of expected utility of final wealth) and financial market structures, see, among many others, Boulier et al. (2001), Haberman and Vigna (2002), Deelstra et al. (2003), Devolder et al. (2003), Battocchio and Menoncin (2004), Cairns et al. (2006) and Di Giacinto et al. (2011).

The long-term investment planning of pension schemes is less frequently cast in the framework of a mean–variance portfolio selection. Mean–variance problems for DC plans are solved in He and Liang (2013), Yao et al. (2013), Vigna (2014), Guan and Liang (2015) and Wu et al. (2015).

The likely reason for the scarcity of literature is the well-known difficulty in solving the mean–variance optimisation problem in both a discrete multi-period framework and in continuous time. The first solution in continuous time to this kind of problem was found in Richardson (1989), and subsequently by Bajeux-Besnainou and Portait (1998), both through the so-called martingale approach. In the first paper, the financial market consists of a riskless and a risky asset, and there is no derivation of the efficient frontier. In the second paper, the interest rate is stochastic, the efficient frontier is derived, and explicit solutions are found in the special case of the Vasiček (1977) model. Li and Ng (2000) and Zhou and Li (2000) solved, respectively in the multiperiod framework and in continuous time, the mean–variance problem by transforming it into a standard stochastic optimal control problem. Since then, a number of extensions have been following.

Even if the choice of the most appropriate point on the efficient frontier is relevant for matching at best investors' preference and, thus, practically implement the mean–variance approach, the literature devotes little attention to this issue. One of the main contributions of this paper is to enhance the comprehension of how

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http://dx.doi.org/10.1016/j.insmatheco.2017.08.002

to select the correct subjective level of risk/reward for a member of a DC pension scheme. We interpret the mean–variance problem as a target-based problem and provide a closed-form one-to-one simple relationship between the target (in terms of final wealth to be reached) and the appropriate level of risk/reward. Furthermore, we show how to keep under control the ruin probability through the choice of the target. Finally, we provide a lowest threshold for the target as a function of the initial wealth and the expected present value of future contributions.

We stress the importance of targets in DC pension funds. Let us consider, for instance, the so-called replacement ratio, i.e. the ratio between the pension rate and the final salary. The achievement of a minimum replacement ratio was guaranteed in DB pension schemes, but not in DC pension schemes. The possibility of selecting a suitable wealth-target at retirement might enable the members of a DC plan to get close to a desired replacement ratio, and might help reducing the inequity among pension fund members belonging to different cohorts that is typical of DC plans (see e.g. Knox, 1993).

The equivalence between mean–variance criterion and the target-based approach is one of the characteristics that make the mean–variance preferences appealing with respect to other types of preferences. Due to this equivalence, the identification of the risk profile for the mean–variance investor can be done via the selection of a final target at retirement, while it is done via the selection of an abstract risk aversion coefficient for other common types of preferences (e.g., the relative risk aversion coefficient for power preferences, the absolute risk aversion coefficient for exponential preferences etc.). For the average pension fund member it is easier to select a wealth target rather than an abstract index. Our selection of mean–variance preferences is also motivated by the evidence that the performance of most investment funds is determined according to mean–variance criteria (see Chiu and Zhou, 2011).

The relationship between targets and points on the mean–variance efficient frontier was introduced by Zhou and Li (2000), and in the context of a DC pension plan was pointed out by Vigna (2014) in a Black and Scholes financial market with constant contribution. In this paper, we extend Vigna (2014) to a more general complete financial market with an arbitrary number of risky assets, risk sources and state variables, and stochastic contribution.

A second contribution of our work is the analytical solution of the mean–variance problem in a DC pension plan in a quite general financial framework. A rather important special case with stochastic interest rate and stochastic salary is solved explicitly and analysed in detail.

As a third contribution, we propose an empirical methodology for implementing the non-negativity constraints on portfolio shares. This issue is usually neglected by the literature on DC pension funds with stochastic interest rate. Furthermore, the comparison between optimal non-constrained strategies and suboptimal constrained strategies is presented. The constrained strategies turn out to be similar to the empirical investment strategies actually adopted by the DC pension schemes in UK.

The remainder of the paper is organised as follows. In Section 2 we outline the general financial market and derive the wealth dynamics. In Section 3 the mean–variance optimisation problem is solved using the embedding technique introduced by Zhou and Li (2000) and the martingale approach; the optimal portfolio, the ruin probability and the efficient frontier are provided analytically. In Section 4 the equivalence between the mean–variance approach and the target-based approach is shown, and guidelines for the practical implementation of the mean–variance model are provided. Section 5 contains a numerical application and presents a special case with financial market consisting in a riskless asset, one bond and one stock. Two stochastic state variables are considered: the riskless interest rate following the Vasiček (1977) dynamics and the contribution following a geometric Brownian motion. The optimal portfolio and the efficient frontier are analysed with different risk profiles, and suboptimal strategies with non-negative weights are introduced and studied. Section 6 concludes. All proofs are gathered in Appendix.

2. The framework

The financial market is arbitrage free, complete, frictionless, and continuously open at any time $t \in [0, T]$. The risk is described by a set of n independent Brownian motions $W(t)$, defined on the complete filtered probability space $\{\Omega, \mathcal{F}(t), \mathbb{P}\}$, where $\mathcal{F}(t)$ is the filtration generated by the Brownian motions and $\mathbb{P}$ is the real-world probability measure. The financial market is described by the following variables:

- s state variables $z(t)$ (with $z(0) = z_0 \in \mathbb{R}^s$ known) whose values solve the stochastic differential equation (SDE)

$$\underset{s\times 1}{dz(t)} = \underset{s\times 1}{\mu_z(t,z)}dt + \underset{s\times n}{\Omega(t,z)}\underset{n\times 1}{dW(t)}; \tag{1}$$

- one riskless asset whose price $G(t)$ solves the (ordinary) differential equation

$$dG(t) = G(t)r(t,z)\,dt,$$

where $r(t,z)$ is the spot instantaneously riskless interest rate;

- n risky assets whose prices $P(t)$ (with $P(0) = P_0 \in \mathbb{R}^n$ known) solve the matrix stochastic differential equation

$$\underset{n\times 1}{dP(t)} = \underset{n\times n}{I_P}\left[\underset{n\times 1}{\mu(t,z)}dt + \underset{n\times n}{\Sigma(t,z)}\underset{n\times 1}{dW(t)}\right], \tag{2}$$

where I_P is the $n \times n$ square diagonal matrix that reports the prices $P_1, P_2, \ldots, P_n$ on the diagonal and zero elsewhere.

Drift and diffusion terms in (1) and (2) are assumed to satisfy the usual conditions for the existence and uniqueness of a strong solution to the SDEs.

The absence of arbitrage and completeness imply the existence of a unique risk-neutral equivalent martingale measure $\mathbb{Q}$. There exists a unique vector of market prices of risk $\xi(t,z)$ which solves the linear system $\Sigma(t,z)\,\xi(t,z) = \mu(t,z) - r(t,z)\,\mathbf{1}$, where $\mathbf{1}$ is a vector of 1's (i.e. $\exists\Sigma(t,z)^{-1}$). Assuming that $\xi(t,z)$ satisfies Novikov's condition, the Girsanov theorem applies and the Wiener processes $dW(t)$ can be rewritten under $\mathbb{Q}$ as follows:

$$dW^{\mathbb{Q}}(t) = \xi(t,z)\,dt + dW(t). \tag{3}$$

The Radon–Nikodym derivative is (the prime denotes transposition):

$$m(t_0,t) = e^{-\frac{1}{2}\int_{t_0}^{t}\xi(u,z)'\xi(u,z)du - \int_{t_0}^{t}\xi(u,z)'dW(u)} \iff \begin{cases} dm(t_0,t) = -m(t_0,t)\,\xi(t,z)'dW(t), \\ m(t_0,t_0) = 1. \end{cases}$$

Thus, given any t-measurable random variable $Z(t)$, the following relationship holds true

$$\mathbb{E}_{t_0}^{\mathbb{Q}}[Z(t)] = \mathbb{E}_{t_0}[Z(t)\cdot m(t_0,t)], \tag{4}$$

where $\mathbb{E}_{t_0}^{\mathbb{Q}}[\bullet]$ and $\mathbb{E}_{t_0}[\bullet]$ are the expected values, under the risk neutral and the real world probabilities respectively, conditional to $\mathcal{F}(t_0)$.

Let $B(t,T)$ be the price in t of a zero-coupon bond expiring in T, and $\sigma_B(t,T)$ the (vector) diffusion term of $\frac{dB(t,T)}{B(t,T)}$. It is well known