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Ordering optimal deductible allocations for stochastic arrangement increasing risks

Chen Li [a], Xiaohu Li [b,*]

[a] School of Science, Tianjin University of Commerce, Tianjin 300134, China

[b] Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken NJ 07030, USA

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ABSTRACT

The insurer usually solicits the insured through granting a certain amount of deductible to multiple risks according to his/her own will. Due to the nonlinear nature of the concerned optimization problem, in the literature on the optimal allocations of deductibles researchers usually assume independence or comonotonicity among concerned risks and ignore the impact due to frequency. In this study we build two sufficient conditions for the decreasing optimal allocation of deductibles, relaxing the stochastic arrangement increasing or right tail weakly stochastic arrangement increasing discount factors in Cai and Wei (2014, Theorems 6.3 and 6.6) to the conditionally upper orthant arrangement increasing or weak conditionally upper orthant arrangement increasing frequencies.

1. Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a vector of n random risks faced by a policyholder, and denote $\boldsymbol{T} = (T_1, \ldots, T_n)$ the vector of the corresponding *occurrence frequencies*. Through paying a premium, the policyholder could obtain a total coverage from an insurer. In the case of a total deductible of $d > 0$ granted for X_i's, let $\boldsymbol{d} = (d_1, \ldots, d_n)$ be an *allocation* vector and denote $\mathcal{A}_d$ all admissible allocation vectors such that $\sum_{i=1}^n d_i = d$ and $d_i \geq 0$ for all $i \in \mathcal{I}_n = \{1, \ldots, n\}$, then the policyholder gets the total discounted retained loss $\sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)$, where $x \wedge d = \min\{x, d\}$ and $\delta \geq 0$ is the discount rate. Let ω be the wealth of the policyholder after the premium is paid and $u(x)$ be his/her utility function, then, from the viewpoint of the risk-averse policyholder, the optimization allocation problem of deductibles is summarized as

$$\begin{cases} \max\limits_{\boldsymbol{d} \in \mathcal{A}_d} \mathsf{E}\left[u\left(\omega - \sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)\right)\right], \\ \text{where } u \text{ is increasing and concave, and } \boldsymbol{X} \text{ is independent of } \boldsymbol{T}. \end{cases} \tag{1.1}$$

Let $\phi(x) = -u(\omega - x)$. Due to the equivalence between increasing and convex ϕ and increasing and concave u, the problem (1.1) can be equivalently rephrased as

$$\begin{cases} \min\limits_{\boldsymbol{d} \in \mathcal{A}_d} \mathsf{E}\left[\phi\left(\sum_{i=1}^n e^{-\delta T_i}(X_i \wedge d_i)\right)\right], \\ \text{where } \phi \text{ is increasing and convex, and } \boldsymbol{X} \text{ is independent of } \boldsymbol{T}. \end{cases}$$

Cheung (2007) and Hua and Cheung (2008a) took the first step to study the above optimization problems. Also, Hua and Cheung (2008b) studied the worst allocations of deductibles from the viewpoint of the insurer, and from the viewpoint of the policyholder Zhuang et al. (2009) derived some new refined results on the ordering of the optimal allocations of deductibles with respect to the family of distortion risk measures. Later, from the view point of policyholder with increasing utility function Lu and Meng (2011), Hu and Wang (2014) and Li and Li (2017) also had a study on the optimal allocation of deductibles for mutually independent risks without frequency impact and arrayed in the sense of likelihood ratio order, hazard rate order or reversed hazard rate order.

Denote by $\boldsymbol{d}^* = (d_1^*, \ldots, d_n^*)$ one solution of (1.1). The earlier research in the literature concerned with this problem either ignores the frequency impact or simply assumes the mutual independence for frequencies of occurrence. For example, Cheung (2007) proved that (i) for mutually independent $\boldsymbol{X}$ and $\delta = 0$ (no frequency impact), $d_i^* \geq d_j^*$ for $1 \leq i \neq j \leq n$ if X_i is smaller than X_j in the hazard rate order, and (ii) for comonotonic $\boldsymbol{X}$ (the worst dependent structure of severities of risks) and $\delta = 0$ (no frequency impact), $d_i^* \geq d_j^*$ for $1 \leq i \neq j \leq n$ if X_i is stochastically smaller

* Corresponding author.
E-mail addresses: mathxhli@hotmail.com, xiaohu.li@stevens.edu (X. Li).

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than X_j. Hua and Cheung (2008a) further showed, for comonotonic $\boldsymbol{X}$ and mutually independent $\boldsymbol{T}$, that $d_i^* \ge d_j^*$ if T_j is smaller than T_i in the likelihood ratio order and X_i is smaller than X_j in the stochastic order, for any $1 \le i \ne j \le n$ and $\delta > 0$. Later, Zhuang et al. (2009) obtained that (i) for mutually independent $\boldsymbol{X}$ and $\boldsymbol{T}$, $d_i^* \ge d_j^*$ if T_j is smaller than T_i in the likelihood ratio order and X_i is smaller than X_j in the likelihood ratio order, for any $1 \le i \ne j \le n$ and $\delta > 0$, and (ii) for comonotonic $\boldsymbol{X}$ and mutually independent $\boldsymbol{T}$, $d_i^* \ge d_j^*$ if T_j is smaller than T_i in the reversed hazard rate order and X_i is smaller than X_j in the stochastic order, for any $1 \le i \ne j \le n$ and $\delta > 0$. Recently, without regard to frequency effect (i.e, $\delta = 0$) Li and You (2015) further proved that $d_i^* \ge d_j^*$ for any $1 \le i < j \le n$ if X_i is smaller than X_j in the hazard rate order and $\boldsymbol{X}$ has an Archimedean survival copula with a log-convex generator.

Along this line, some effort had been put into the optimization problem (1.1) in this decade. One remarkable improvement is due to the introduction of statistical dependence into either severities $\boldsymbol{X}$ or frequencies of occurrence $\boldsymbol{T}$, which makes the model more flexible in practice and more general in theoretical sense. For comonotonic $\boldsymbol{X}$ with $X_1 \le_{\rm st} \cdots \le_{\rm st} X_n$ and $\boldsymbol{T}$ with $T_1 \ge_{\rm lr} \cdots \ge_{\rm lr} T_n$ and coupled by an Archimedean copula, Li and You (2012, Theorem 2) showed that the allocation $\boldsymbol{d}$ with $d_1 \le \cdots \le d_n$ is one of least favorable allocations for the policyholder. Denote $\boldsymbol{W} = (W_1, \ldots, W_n) = (e^{-\delta T_1}, \ldots, e^{-\delta T_n})$ the vector of *discount factors* corresponding to frequencies of occurrence $\boldsymbol{T}$. Recently, Cai and Wei (2014, Theorems 6.3 and 6.6) developed two sufficient conditions for $d_1^* \ge \cdots \ge d_n^*$: (i) $\boldsymbol{X}$ and $\boldsymbol{W}$ both SAI; (ii) comonotonic $\boldsymbol{X}$ with $X_1 \le_{\rm st} \cdots \le_{\rm st} X_n$ and RWSAI $\boldsymbol{W}$. This study further builds the following two new sufficient conditions for $d_1^* \ge \cdots \ge d_n^*$, relaxing the dependence on $\boldsymbol{W}$ at a cost of adding some restrictions on the utility functions:

- Nonnegative SAI $\boldsymbol{X}$, CUOAI $\boldsymbol{W}$, and u with convex u' and concave u'';
- Nonnegative and comonotone $\boldsymbol{X}$ with $X_1 \le_{\rm st} \cdots \le_{\rm st} X_n$, WCUOAI $\boldsymbol{W}$, and u with convex u' and concave u''.

The rest of this paper is rolled out as follows: Section 2 introduces some preliminaries including concerned stochastic orders, various stochastic versions of the arrangement increasing properties as well as several useful lemmas. In Section 3 we build the two sufficient conditions for the decreasing optimal allocation of deductibles. Also, two numerical examples are presented in Section 4 to illustrating the main results. All proofs for technical lemmas are given in Section 5.

From now on, we denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+^2 = (0, +\infty)^2$ and let $x \vee y = \max\{x, y\}$. Throughout this note, expectations are implicitly assumed finite whenever utilized, and the terms *increasing* and *decreasing* mean *nondecreasing* and *nonincreasing*, respectively.

2. Some preliminaries

For two real-valued vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$, let the permutation $\tau_{i,j}(\boldsymbol{x}) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)$ and the sub-vector $\boldsymbol{x}_{(i,j)} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ for $1 \le i < j \le n$, and denote the inner product $\boldsymbol{\lambda} \cdot \boldsymbol{x} = \lambda_1 x_1 + \cdots + \lambda_n x_n$ and the pairwise minimum $\boldsymbol{\lambda} \wedge \boldsymbol{x} = (\lambda_1 \wedge x_1, \ldots, \lambda_n \wedge x_n)$.

A random variable X is said to be smaller than the other one Y in the

(i) *usual stochastic order* (denoted as $X \le_{\rm st} Y$) if $\mathrm{P}(X > x) \le \mathrm{P}(Y > x)$ for all x;
(ii) *increasing convex order* (denoted as $X \le_{\rm icx} Y$) if $\int_x^{+\infty} \mathrm{P}(X > t)\,\mathrm{d}t \le \int_x^{+\infty} \mathrm{P}(Y > t)\,\mathrm{d}t$ for all x, provided the existence of the two integrals;
(iii) *mean residual life order* (denoted as $X \le_{\rm mrl} Y$) if $\frac{\int_x^{+\infty} \mathrm{P}(X>t)\,\mathrm{d}t}{\mathrm{P}(X>x)} \le \frac{\int_x^{+\infty} \mathrm{P}(Y>t)\,\mathrm{d}t}{\mathrm{P}(Y>x)}$ for all x.

For stochastic orders one may refer to Shaked and Shanthikumar (2007), and Li and Li (2013).

A real function $f(\boldsymbol{x})$ is said to be *arrangement increasing* (AI) if $f(\boldsymbol{x}) \ge f\big(\tau_{i,j}(\boldsymbol{x})\big)$ for any $\boldsymbol{x}$ with $x_i \le x_j$ and $1 \le i < j \le n$. For $i \ne j$, let $\Delta_{ij} g(\boldsymbol{x}) = g(\boldsymbol{x}) - g\big(\tau_{i,j}(\boldsymbol{x})\big)$ and denote

$$\mathcal{A}_{\rm s}^{i,j}(n) = \big\{g(\boldsymbol{x}) : \Delta_{ij} g(\boldsymbol{x}) \ge 0 \text{ for any } x_j \ge x_i\big\},$$
$$\mathcal{A}_{\rm rw}^{i,j}(n) = \big\{g(\boldsymbol{x}) : \Delta_{ij} g(\boldsymbol{x}) \text{ is increasing in } x_j \in [x_i, \infty) \text{ for any } x_i\big\}.$$

According to Cai and Wei (2014), a random vector $\boldsymbol{X}$ on $\mathbb{R}^n$ is said to be

(i) *stochastic arrangement increasing* (SAI) if $\mathrm{E}[g(\boldsymbol{X})] \ge \mathrm{E}\big[g\big(\tau_{i,j}(\boldsymbol{X})\big)\big]$ for any $g \in \mathcal{A}_{\rm s}^{i,j}(n)$ and $1 \le i < j \le n$ such that the expectations exist;
(ii) *right tail weakly stochastic arrangement increasing* (RWSAI) if $\mathrm{E}[g(\boldsymbol{X})] \ge \mathrm{E}\big[g\big(\tau_{i,j}(\boldsymbol{X})\big)\big]$ for any $g \in \mathcal{A}_{\rm rw}^{i,j}(n)$ and $1 \le i < j \le n$ such that the expectations exist.

For absolutely continuous random vectors, the SAI property is equivalent to the corresponding AI probability density (see Cai and Wei, 2014), and the RWSAI property is equivalent to the upper tail permutation decreasing (UTPD) probability density (see Li and You, 2015). Cai and Wei (2014) and Li and Li (2016) also introduced the following weak versions. $\boldsymbol{X}$ is said to be

(i) *upper orthant arrangement increasing* (UOAI) if the joint survival function $\bar{F}(\boldsymbol{x})$ is AI;
(ii) *conditionally upper orthant arrangement increasing* (CUOAI) if $[(X_i, X_j) \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}]$ is UOAI for any fixed $\boldsymbol{x}_{(i,j)}$ in support of $\boldsymbol{X}_{(i,j)}$ and any $1 \le i < j \le n$;
(iii) *lower orthant arrangement increasing* (LOAI) if the joint distribution function $F(\boldsymbol{x})$ is AI;
(iv) *conditionally lower orthant arrangement increasing* (CLOAI) if $[(X_i, X_j) \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}]$ is LOAI for any $\boldsymbol{x}_{(i,j)}$ in support of $\boldsymbol{X}_{(i,j)}$ and any $1 \le i < j \le n$;
(v) *weak conditionally lower orthant arrangement increasing* (WCLOAI) if
$$\int_{-\infty}^{t} \mathrm{P}\big(X_i \le x_i, X_j \le x_j \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}\big)\,\mathrm{d}x_i \ge \int_{-\infty}^{t} \mathrm{P}\big(X_i \le x_j, X_j \le x_i \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}\big)\,\mathrm{d}x_i,$$
for all $1 \le i < j \le n$, $t \le x_j$ and any $\boldsymbol{x}_{(i,j)}$ in support of $\boldsymbol{X}_{(i,j)}$.

Here we introduce the following dependence notion as the dual of WCLOAI.

Definition 2.1. A random vector $\boldsymbol{X}$ is said to be *weak conditionally upper orthant arrangement increasing* (WCUOAI) if, for all $1 \le i < j \le n$, $t \ge x_j$ and any $\boldsymbol{x}_{(i,j)}$ in support of $\boldsymbol{X}_{(i,j)}$,

$$\int_{t}^{+\infty} \mathrm{P}\big(X_i > x_i, X_j > x_j \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}\big)\,\mathrm{d}x_i \le \int_{t}^{+\infty} \mathrm{P}\big(X_i > x_j, X_j > x_i \mid \boldsymbol{X}_{(i,j)} = \boldsymbol{x}_{(i,j)}\big)\,\mathrm{d}x_i.$$

Based on (4.2) of Cai and Wei (2014), one can easily verify the following chain of implications,

$$\text{SAI} \Longrightarrow \text{RWSAI} \Longrightarrow \text{CUOAI} \Longrightarrow \text{WCUOAI} \Longrightarrow X_1^{\perp} \le_{\rm icx} \cdots \le_{\rm icx} X_n^{\perp},$$

where $(X_1^{\perp}, \ldots, X_n^{\perp})$ is the independence version of $\boldsymbol{X}$. As per the next numerical example, neither CUOAI imply RWSAI nor WCUOAI imply CUOAI.