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A revisit to ruin probabilities in the presence of heavy-tailed insurance and financial risks

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ABSTRACT

Recently, Sun and Wei (2014) studied the finite-time ruin probability under a discrete-time insurance risk model, in which the one-period insurance and financial risks are assumed to be independent and identically distributed copies of a random pair (X, Y). For the heavy-tailed case, under a restriction on the dependence structure of (X, Y), they established an asymptotic formula for the finite-time ruin probability. In this paper we make an effort to remove this restriction as it excludes the cases with asymptotically dependent X and Y. We also extend the study to the infinite-time ruin probability. Employing a multivariate regular variation framework, we simplify the formula so that it shows in a transparent way how the ruin probabilities are affected by the tail dependence of (X, Y).

1. Introduction

In this paper we consider an insolvency problem for an insurer who makes risky investments. We model the insurance business and the investments in a discrete-time model. Suppose that at time 0 the insurer holds an initial capital of amount x. At the beginning of each period k, the insurer invests its current wealth into a portfolio of assets that provides an overall return rate of $R_k \in (-1, \infty)$. During each period k, the insurer's realized net profit from the insurance business is denoted by a random variable $Z_k \in \mathbb{R}$, which is roughly equal to premiums collected minus insurance claims paid and other expenses incurred.

Denote by W_k the wealth of the insurer at k, $k \in \mathbb{N}$. The insurance business and investments together lead to an evolution of the wealth process as follows:

$$W_k = (1 + R_k)W_{k-1} + Z_k, \quad k \in \mathbb{N}, \tag{1.1}$$

where $W_0 = x$. Iterating (1.1) yields

$$W_k = x\prod_{j=1}^{k}(1 + R_j) + \sum_{i=1}^{k} Z_i \prod_{j=i+1}^{k}(1 + R_j),$$

where, and throughout this paper, multiplication over an empty index set produces value 1 by convention. We consider the probabilities of ruin in both finite time and infinite time, which are given by

$$\psi(x; n) = P\left(\min_{0\le k\le n} W_k < 0 \middle| W_0 = x\right), \quad n \in \mathbb{N} \cup \{\infty\},$$

where for $n = \infty$ we understand $0 \le k \le n$ as $0 \le k < \infty$.

The discrete-time risk model and the ruin probabilities described above were first introduced by Tang and Tsitsiashvili (2003, 2004), and were thereafter studied extensively in insurance mathematics and applied probability; see, e.g., Chen and Ng (2007), Chen (2011), Fougères and Mercadier (2012), Yang and Wang (2013), and Li and Tang (2015). Continuous-time versions of the study have also appeared in the literature; see Norberg (1999), Klüppelberg and Kostadinova (2008), Paulsen (2008), Heyde and Wang (2009), Bankovsky et al. (2011) and Hult and Lindskog (2011), among

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others. Generally, the importance of ruin theory lies in its guidance to insurers and regulators for risk capital calculation, as well as its application to the pricing of related insurance products. For example, to comply with regulation frameworks such as EU Solvency II, insurers have to hold enough risk capital so that the probability of ruin is sufficiently low. In addition, some insurance-linked securities such as contingent convertibles may use insurers' probability of ruin as a trigger. Both cases will require an assessment of the probability of ruin, making the notion of ruin theory relevant.

For ease of presentation, let us introduce

$$X_i = -Z_i \quad \text{and} \quad Y_i = \frac{1}{1+R_i}, \quad i \in \mathbb{N},$$

where $X_i \in \mathbb{R}$ quantifies the insurance risk and $Y_i \in [0, \infty)$ quantifies the financial risk. Then, for each $n \in \mathbb{N} \cup \{\infty\}$, we can rewrite the probability of ruin as

$$\psi(x; n) = P\left(\min_{0\le k\le n}\left(x\prod_{j=1}^{k}(1+R_j) + \sum_{i=1}^{k} Z_i \prod_{j=i+1}^{k}(1+R_j)\right) < 0\right)$$
$$= P\left(\max_{1\le k\le n}\sum_{i=1}^{k} X_i \Pi_i > x\right),$$

where $\Pi_i = \prod_{j=1}^{i} Y_j$ denotes the stochastic discount factor over the first i periods, $i \in \mathbb{N}$.

A recent trend of related studies is to describe the impact of the dependence between the insurance and financial risks on the asymptotic behavior of the ruin probability. In particular, Chen (2011) studied the finite-time ruin probability, assuming that the insurance and financial risks (X_i, Y_i), $i \in \mathbb{N}$, are described by independent and identically distributed (i.i.d.) random pairs with common bivariate Farlie–Gumbel–Morgenstern distribution. Fougères and Mercadier (2012) carried on their study under a general multivariate regular variation (MRV) framework for the insurance and financial risks. Tang and Yuan (2012) introduced an algorithm for the computation of ruin probability, under auto-regressive models for both one-period claim amounts and one-period log-return rates.

In particular, Sun and Wei (2014) moved the study forward by considering a general class of heavy-tailed distributions for X_i and Y_i, $i \in \mathbb{N}$, and a weak within-period dependence structure between the two kinds of risks described by (3.1). However, as shown by Lemma 3.1, their assumption of dependence is quite restrictive and rules out the cases where X_i and Y_i are asymptotically dependent. This paper is a significant continuation to this line of study, where we remove the assumption on the dependence structure needed by Sun and Wei (2014) for their result. The price we pay for this extension is a slightly more restrictive assumption on the distribution of X_iY_i. In addition, we also extend the study to the infinite-time horizon.

The rest of the paper is organized as follows. In Section 2 we introduce some preliminaries needed for our study, in Section 3 we present our main results, and in Section 4 we provide some refinements of the results, which also show the impact of the tail dependence between the insurance and financial risks on the ruin probabilities.

2. Preliminaries

Let us first introduce some notational conventions. Throughout this paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup f(x)/g(x) \le 1$, write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$, and write $f(x) \asymp g(x)$ if $f(\cdot)$ and $g(\cdot)$ are weakly equivalent, that is, $0 < \liminf f(x)/g(x) \le \limsup f(x)/g(x) < \infty$.

For two real numbers x and y, write $x \vee y = \max\{x, y\}$. For a non-decreasing function h, denote by $h^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : h(x) \ge y\}$ its generalized inverse function. For a distribution function H and every $0 \le y \le 1$, by relations (0.6b) and (0.6c) of Resnick (1987) we have

$$H(H^{\leftarrow}(y)) \ge y, \quad \text{and} \quad H^{\leftarrow}(y) \le x \quad \text{if and only if } y \le H(x). \tag{2.1}$$

For a random variable ξ distributed by H, write

$$b_\xi(x) = \left(1/\overline{H}\right)^{\leftarrow}(x) = H^{\leftarrow}(1 - 1/x). \tag{2.2}$$

Next, we recall some concepts of heavy-tailed distributions. A distribution function H on $\mathbb{R}$ is said to be long tailed, written as $H \in \mathcal{L}$, if it has an ultimate right tail (that is, $\overline{H}(x) > 0$ for all $x \in \mathbb{R}$) and it holds that

$$\lim_{x\to\infty} \frac{\overline{H}(x+y)}{\overline{H}(x)} = 1 \quad \text{for all } y \in \mathbb{R}.$$

This implies that there is some positive function $a(\cdot)$, with $a(x) \uparrow \infty$ and $a(x) = o(x)$, such that

$$\overline{H}(x + a(x)) \sim \overline{H}(x).$$

A distribution function H on $\mathbb{R}$ is said to be dominatedly varying tailed, written as $H \in \mathcal{D}$, if it has an ultimate right tail $\overline{H}$ satisfying

$$\limsup_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} < \infty \quad \text{for all } 0 < y < 1.$$

The intersection $\mathcal{L} \cap \mathcal{D}$ is a class of heavy-tailed distributions with wide application; see, e.g., Geluk and Tang (2009) and Sun and Wei (2014) for related discussions.

An important subclass of $\mathcal{L} \cap \mathcal{D}$ is the class $\mathcal{C}$, which is large enough to contain almost all practically useful distribution functions in $\mathcal{L} \cap \mathcal{D}$. A distribution function H on $\mathbb{R}$ is said to be in the class $\mathcal{C}$, written as $H \in \mathcal{C}$, if it has a consistently varying tail; that is,

$$\lim_{y\uparrow 1}\limsup_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} = 1, \quad \text{or, equivalently,}$$
$$\lim_{y\downarrow 1}\liminf_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} = 1. \tag{2.3}$$

Clearly, the class $\mathcal{C}$ contains the famous class $\mathcal{R}$ as a proper subset. A distribution function H on $\mathbb{R}$ is said to be in the class $\mathcal{R}$, written as $H \in \mathcal{R}$, if it has a regularly varying tail; that is, if for some $\gamma > 0$,

$$\lim_{x\to\infty} \frac{\overline{H}(xy)}{\overline{H}(x)} = y^{-\gamma} \quad \text{for all } y > 0.$$

Furthermore, if the above holds, then we say that the tail $\overline{H}$ is regularly varying with index $-\gamma$, denoted as $\overline{H} \in \mathrm{RV}_{-\gamma}$. We refer the reader to Bingham et al. (1987), Resnick (1987), and Embrechts et al. (1997) for comprehensive treatments of regular variation and heavy-tailed distributions.

Finally, we introduce the lower and upper Matuszewska indices of distribution functions, which are very useful in describing their tail behavior. For a distribution function H on $\mathbb{R}$ with an ultimate right tail, its lower and upper Matuszewska indices are defined by

$$M_* = \sup\left\{-\frac{\log \overline{H}^*(y)}{\log y} : y > 1\right\} \quad \text{and}$$
$$M^* = \inf\left\{-\frac{\log \overline{H}_*(y)}{\log y} : y > 1\right\},$$