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# Optimal dividend payout model with risk sensitive preferences

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ABSTRACT

We consider a discrete-time dividend payout problem with risk sensitive shareholders. It is assumed that they are equipped with a risk aversion coefficient and construct their discounted payoff with the help of the exponential premium principle. This leads to a risk adjusted discounted cash flow of dividends. Within such a framework not only the expected value of the dividends is taken into account but also their variability. Our approach is motivated by a remark in Gerber and Shiu (2004). We deal with the finite and infinite time horizon problems and prove that, even in this general setting, the optimal dividend policy is a band policy. We also show that the policy improvement algorithm can be used to obtain the optimal policy and the corresponding value function. Next, an explicit example is provided, in which the optimal policy is shown to be of a barrier type. Finally, we present some numerical studies and discuss the influence of the risk sensitive parameter on the optimal dividend policy.



## 1. Introduction

The dividend payout model in risk theory is a classical problem that was introduced by de Finetti (1957). Since then there have been various extensions. The goal is to find for the free surplus process of an insurance company, a dividend payout strategy that maximises the expected discounted dividends until ruin. Typical models for the surplus process are compound Poisson processes, diffusion processes, general renewal processes or discrete time processes. The reader is referred to Albrecher and Thonhauser (2009) and Avanzi (2009), where excellent overviews of recent results are provided.

Up to now most of the research has been done for the risk neutral perspective, where the expected discounted dividends until ruin are considered. Obviously this criterion does not take the variability of the dividends into account. From the shareholders' perspective or from an economic point of view it would be certainly desirable to reduce the variability of the dividends. Risk should be incorporated in any kind of economic decision and shareholders are in general risk averse. In Gerber and Shiu (2004) the authors propose the problem of maximising the expected *utility* of discounted dividends until ruin instead. Such a criterion is able to model risk aversion. In Grandits et al. (2007) the authors consider the dividend problem with an exponential utility in a diffusion setting. They show under some assumptions that there is a time dependent optimal barrier. Bäuerle and Jaśkiewicz (2015) consider a discrete time setting and prove the optimality of a band policy for the exponential utility and partly characterise the optimal dividend policy in a power utility setting. To the best of our knowledge these are so far the only papers dealing with risk sensitive dividend problems.

In this paper, we treat now the discrete time setting with state space $\mathbb{R}_+$ like in Albrecher et al. (2011) and Socha (2015). However, we propose a new approach, where we consider risk sensitive preferences. Namely, the risk adjusted discounted cash flow of the shareholder is now of the form

$$V_t = \alpha_t + M(V_{t+1}), \quad \text{where } M(V_{t+1}) = -\frac{\beta}{\gamma}\ln\left(\mathbb{E}_t e^{-\gamma V_{t+1}}\right),$$

$\alpha_t$ is the dividend paid at time $t$, $\beta \in (0,1)$ is a discount factor, $\gamma > 0$ is the risk sensitive parameter and $V_t$ is the risk adjusted discounted cash flow of dividends from time $t$ onwards. These preferences are not time additive in the future dividends anymore and allow to model risk aversion. Note that we are here concerned about the variability of each dividend paid. This is in contrast to Grandits et al. (2007) and Bäuerle and Jaśkiewicz (2015), where the utility of the total discounted dividends is considered. For the exponential utility with discount factor 1 both approaches are equivalent.

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The risk sensitive preferences considered in this paper belong to a wider class of recursive preferences studied extensively in macroeconomics and finance. They enjoy attention, because they allow to disentangle risk attitudes from intertemporal substitution. In particular, Epstein and Zin (1989) and Weil (1990) motivated the use of the certainty equivalent $M$. Its concavity (see footnote in Section 2) would amplify risk aversion above intertemporal substitution. Furthermore, the concavity of $M$ would cause the agent to prefer the early resolution of uncertainty (see Kreps and Porteus, 1978). The aforementioned recursive preference functional is still analytically tractable and retains the main behaviour features of the risk neutral case with $M$ replaced by the expectation operator. One of the first papers on optimal control with this risk adjusted certainty equivalent in discrete time is Hansen and Sargent (1995). It considers special LQ-problems. In recent years there is a growing number of papers that study various model aspects with certainty equivalents, see for instance, Anderson (2005), Tallarini (2000) and Weil (1993).

Our model can be viewed as a Markov decision process with specific transition probability and payoff functions. Therefore, it is worth mentioning that Markov decision processes with dynamic risk maps and discounted costs were examined by Ruszczyński (2010). However, his results do not imply ours, since he studied bounded cost functions and coherent risk measures. In particular, such a risk measure must be positively homogeneous. Further, Shen et al. (2013) generalise the paper of Ruszczyński (2010) to unbounded gains and the risk sensitive average reward case. However, in their approach they apply the weighted norm approach, which results in rather stringent assumptions. Moreover, they do not analyse the properties of an optimal policy. This analysis, in our case, is necessary to show that the optimal policy has a band structure. Bäuerle and Rieder (2014) considered general certainty equivalents for the accumulated discounted payoffs. All the aforementioned papers deal with Bellman equations and discuss existence and uniqueness of solutions as well as optimal policies. However, their results are not helpful in our special setting.

The main contributions of our paper are threefold. First we are able to give a mathematically rigorous solution technique for these risk sensitive dividend problems over a finite and an infinite time horizon. More precisely, we formulate a Bellman equation which allows to compute the value function over a finite time horizon. We also show that these value functions monotonically approximate the value function of the infinite horizon problem. The infinite horizon value function is also characterised as a fixed point of an operator on a certain set of functions. Second we prove that a stationary optimal policy has a band structure. Hence, even in this more complicated risk sensitive setting, we are able to confirm the same form of optimal dividend payout strategy as in the risk neutral case (for the risk neutral model consult, e.g., Miyasawa, 1962; Morrill, 1966; Gerber, 1974; Borch, 1982). Third we show that the policy improvement algorithm is another feasible way to compute the value function and the optimal dividend payout policy for the infinite time horizon. Finally, we give some numerical examples that shed some light on the optimal policy. For a risk sensitive model with left-sided exponential distribution for the increments of the risk reserve, we show under some assumptions on the parameters that a barrier policy is optimal. This result generalises Socha (2015). For a risk sensitive model with the double-exponential distribution for the increments of the risk reserve, we compute the optimal policy for a three-stage model explicitly. We can see some surprising dependence of the barrier on the risk sensitive parameter.

The paper is organised as follows. In Section 2, we introduce the model and our notation. The finite horizon problem is then considered in Section 3 and the limit to the infinite horizon is discussed in Section 4. In Section 5, we characterise the value function as the unique fixed point of some operator within a certain class of functions. Next we show in Section 6 that an optimal dividend policy in this risk sensitive setting is a band policy. Afterwards we prove the validity of the policy improvement algorithm in this risk sensitive case. In Section 8 we consider an example with left-sided exponential distribution for the increments of the risk reserve and show that a barrier policy is optimal. In Section 9, we provide two examples, where we compute the optimal risk sensitive dividend payout over a time horizon of three and discuss the influence of the risk sensitive parameter on an optimal policy.

## 2. The model

We consider the classical dividend payout problem with risk sensitive recursive evaluation of the dividends, which are paid at discrete times, say $n \in \mathbb{N} := 1, 2, \ldots$. Assume there is an initial surplus $x_1$ and usually $x_1 = x \in \mathbb{R}_+ := [0, +\infty)$. Let $Z_n$ be the difference between premium income and claim size in the $n$th time interval and assume that $Z_1, Z_2, \ldots$ are independent and identically distributed random variables with distribution $\nu$ on $\mathbb{R}$. At the beginning of each time interval the insurer can decide upon paying a dividend. The dividend payment at time $n$ is denoted by $a_n$. If the current risk reserve at time $n \in \mathbb{N}$, say $x_n$, is non-negative, then $a_n$ has to be non-negative and less than or equal to $x_n$. If $x_n < 0$, then the company is ruined and no further dividend can be paid. Hence, the set of admissible dividends is $\mathbb{A}(x_n) := [0, x_n]$, if $x_n \geq 0$ and $\mathbb{A}(x_n) := \{0\}$, if $x_n < 0$. The evolution of the surplus is given by the following equation $x_{n+1} := f(x_n, a_n, Z_n)$, where

$$f(x_n, a_n, Z_n) := \begin{cases} x_n - a_n + Z_n, & \text{if } x_n \geq 0 \\ x_n, & \text{if } x_n < 0. \end{cases}$$

For any $n \in \mathbb{N}$, by $H_n$ we denote the set of all feasible histories of the process up to time $n$, i.e.,

$$h_n := \begin{cases} x_1, & \text{if } n = 1 \\ (x_1, a_1, x_2, \ldots, x_n), & \text{if } n \geq 2, \end{cases}$$

where $a_k \in \mathbb{A}(x_k)$ for $k \in \mathbb{N}$. A *dividend policy* $\pi = (\pi_n)_{n\in\mathbb{N}}$ is a sequence of Borel measurable decision rules $\pi_n : H_n \mapsto \mathbb{R}_+$ such that $\pi_n(h_n) \in \mathbb{A}(x_n)$. Let $\Lambda$ be the set of all real-valued Borel measurable mappings such that $\alpha(x) \in \mathbb{A}(x)$ for every $x \in \mathbb{R}$. A policy $\pi = (\pi_n)_{n\in\mathbb{N}}$ is called Markov, if $\pi_n(h_n) = \alpha_n(x_n)$ for some $\alpha_n \in \Lambda$, every $h_n \in H_n$ and $n \in \mathbb{N}$. A Markov policy is stationary, if $\alpha_n = \alpha$ for some $\alpha \in \Lambda$ and all $n \in \mathbb{N}$. In this case, we write $\pi = \alpha^\infty$. The sets of all policies, all Markov policies, all stationary policies are denoted by $\Pi$, $\Pi^M$ and $\Pi^S$, respectively.

Ruin occurs as soon as the surplus gets negative. The epoch $\tau$ of ruin is defined as the smallest positive integer $n$ such that $x_n < 0$. The question arises as to how the risk sensitive insurance company will choose its dividend strategy to maximise the gain of the shareholder. In this paper, we shall consider the risk adjusted discounted cash flow of dividends in the finite and infinite time horizon, derived with the aid of the *entropic risk measure* also known as the *exponential premium principle.*

Let $X$ be a non-negative real-valued random variable with distribution $\mu$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The entropic risk measure $\rho$ for $X$ is defined as follows

$$\rho(X) = -\frac{1}{\gamma} \ln\Big(\int_{\mathbb{R}_+} e^{-\gamma x}\mu(dx)\Big),$$

where $\gamma > 0$ is a *risk sensitivity parameter* known also as a risk coefficient. Let $Y$ be also a non-negative random variable defined