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Optimal investment strategies for participating contracts

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HIGHLIGHTS

- Closed-form optimal investment strategies are established for managing participating contracts.
- Optimal strategies are derived under a utility maximization framework with an S-shaped utility function.
- A concavification technique and a pointwise optimization procedure are adopted with a martingale approach for the exploration of closed-form optimal solutions.
- A numerical procedure is applied for optimal solutions when investment strategies are constrained by an upper bound.

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ABSTRACT

Participating contracts are popular insurance policies, in which the payoff to a policyholder is linked to the performance of a portfolio managed by the insurer. We consider the portfolio selection problem of an insurer that offers participating contracts and has an S-shaped utility function. Applying the martingale approach, closed-form solutions are obtained. The resulting optimal strategies are compared with portfolio insurance hedging strategies (CPPI and OBPI). We also study numerical solutions of the portfolio selection problem with constraints on the portfolio weights.

1. Introduction

We study the continuous time portfolio selection problem for insurance companies managing the portfolios supporting participating insurance contracts. Participating contracts are constructed to allow policyholders to share in the profits of the investment portfolio, while simultaneously receiving a guarantee that limits their downside. The policyholders pay premiums to the insurer and the collected premiums are pooled within the insurance company's *general account*. The contract payoffs are linked to the performance of this account. The insurance company manages the fund in order to hedge its liabilities, and maximize the performance of its residual share of the portfolio after the liabilities have been paid.

The objective of the present paper is to develop optimal asset management strategies for the insurance companies, whereas most of the existing literature focuses either on the pricing aspect of participating contracts or certain characterization of the risk which the insurance companies are exposed to from writing these contracts. For example, Briys and De Varenne (1994) derive a closed-form valuation based on an option pricing approach for the participating contract, where the policyholder receives a guaranteed rate of interest (namely *point-to-point basis guarantee*) and some bonuses determined as a fraction of financial gains at the maturity of the contract. Other work on pricing includes Grosen and Jørgensen (2002), Siu (2005), and Fard and Siu (2013). The literature that focuses on the characterization of insurance companies' risk exposure includes Kling et al. (2007), Gatzert and Kling (2007), and Bernard and Le Courtois (2012), among others. Kling et al. (2007), and Gatzert and Kling (2007) investigate some standard risk measures of the participating contracts known as *cliquet-style guarantees*, for which the policyholder is credited with a certain rate of return every year. Bernard and Le Courtois (2012) study the resulting risk profile of both the insurance company

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and policyholders under two well-known portfolio insurance strategies (i.e., CPPI and OBPI). Earlier work on asset and liability management for participating contracts has often focused on the problem in discrete time with a finite scenario set. The advantage of this setting is that it allows one to consider more complex and flexible contract structures. Its disadvantages include a lack of closed form solutions, and computational challenges in generating and working with scenario trees. Examples include Consiglio et al. (2008) and Consiglio et al. (2006), both of which employ scenario optimization in discrete time to analyze problems faced by insurers offering participating contracts with minimum guarantees. For a general stochastic control formulation of the problem faced by an insurer maximizing expected utility of the surplus of assets net of liabilities, see Rudolf and Ziemba (2004).

Utility based portfolio selection problems have been intensively studied in the literature on mathematical finance and economics; see, for example, Cvitanić and Karatzas (1992), Karatzas et al. (1991) and Karatzas and Shreve (1998). Our problem differs due to the inclusion of a liability consisting of a participating contract in the investment portfolio. Moreover, decision-makers are taken to be risk averse with respect to gains and risk seeking with respect to losses, which results in an *S-shaped* power utility function. This utility function is exploited in our problem to reflect this behavioral perspective for the insurance company, which plays the role of the asset manager, to derive explicit optimal investment strategies for two participating contracts with *point-to-point basis guarantees*, which we call (following Bernard et al., 2010) *the defaultable participating contract* and *the fully protected participating contract*. The solutions provide insights for the insurance company in constructing portfolios to serve its purposes.

Our derivation of the optimal solutions relies on a combination of a martingale approach and a pointwise optimization technique. The legitimacy of the martingale approach follows from the completeness of the market model we consider. The approach entails determining the best terminal portfolio value and recovering the dynamic investment strategies from this payoff. In the pointwise optimization procedure, we adopt a concavification technique, which has been used by Carpenter (2000) and later by He and Kou (2016).

As we previously noted, one payoff function we consider in this paper is based on a *point-to-point basis guarantee*, following Briys and De Varenne (1994), and its shape is similar to that of the first-loss fee scheme for hedge funds studied by He and Kou (2016). However, in our problem the positive payoff for the insurance company consists of two pieces with a kink point, while in He and Kou (2016) the positive part of payoff is smooth without any kink. Therefore, the use of an *S-shaped* utility function in our problem sets results in an objective function different from that considered by He and Kou (2016). Moreover He and Kou (2016) consider a liquidation barrier for the fund. When the portfolio drops below this boundary, the fund is liquidated immediately. In contrast, we do not employ a liquidation barrier. These problem characteristics significantly complicate the analysis, and the final form of the optimal solutions.

The completeness of the financial market is a key assumption for our derivation of explicit optimal solutions by the martingale approach. In practice, however, regulatory requirements aimed at controlling solvency risk may prevent the insurance company from investing more than a certain fraction of total wealth in the risky assets. In the presence of such regulatory restrictions, the market is no longer complete for the insurance company, and analytical solutions of the control problem are in general no longer attainable. In this paper, we resort to a numerical procedure to compute the optimal solutions in the constrained case to facilitate comparison with the solutions derived by the martingale approach for the unconstrained case.

The remainder of the paper is structured as follows. Section 2 describes participating contracts and presents the formulation of the stochastic control problem. Auxiliary problem formulations are also given in this section. In Section 3, we solve the auxiliary problems using Lagrangian duality and the pointwise optimization technique. The justification for the concavification technique is included in this section as well. Section 4 presents the optimal portfolio value processes and optimal trading strategies for the stochastic control problems. Section 5 presents numerical examples for the solutions from Section 4. In Section 6, we consider the constrained portfolio problem with bounded control. Section 7 provides further discussion and concludes the paper.

2. Participating contracts and problem formulation

2.1. Basics of participating contracts

Let L_0 be the policyholder's total contribution and α be the initial liability-to-asset ratio of the insurer so that the initial capital in the insurer's *general account* is $x_0 := L_0/\alpha > 0$.

We assume that the capital in the general account is invested in a risky asset S and a risk-free bond B with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t = \mu S_t dt + \sigma S_t dW_t, \end{cases}$$

where r is the risk-free rate, $\mu > r$ is the growth rate of the risky asset, $\sigma > 0$ is the volatility, and $W := \{W_t, t \geq 0\}$ is a standard Brownian motion under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbf{F} := \{\mathcal{F}_t, t \geq 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$ of the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let π_t denote the amount of capital invested in the risky asset S at time t, $t \geq 0$. With a trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$, the total portfolio value process, denoted by X_t^π, evolves as follows:

$$dX_t^\pi = [rX_t^\pi + \pi_t(\mu - r)]dt + \sigma \pi_t dW_t. \quad (1)$$

It is natural to assume that the trading strategy π is $\mathbf{F}$-progressively measurable and satisfies $\int_0^T \pi_t^2 dt < \infty$ a.s., which guarantees the existence and uniqueness of strong solution to (1).

The terminal portfolio value X_T^π is shared between the policyholder and the insurer according to a pre-described scheme with certain guarantee features in favor of the policyholder. Below, we introduce two participating contracts with terminal guarantees: (1) a *defaultable participating contract*; and (2) a *fully protected participating contract*. In both contracts, the policyholder is guaranteed a minimum growth rate g (see Briys and De Varenne, 1994) and the guaranteed amount at maturity time T is $L_T^g = L_0 e^{gT}$, where L_0 is the initial liability of the insurer. g is set lower than the risk-free rate.

In the defaultable participating contract, the payoff to the policyholder is given as follows:

$$\begin{aligned} \Theta(X_T^\pi) &= L_T^g + \delta(\alpha X_T^\pi - L_T^g)_+ - (L_T^g - X_T)_+ \\ &= \begin{cases} X_T^\pi, & X_T^\pi < L_T^g, \\ L_T^g, & L_T^g \leq X_T^\pi \leq \dfrac{L_T^g}{\alpha}, \\ \delta\alpha X_T^\pi + (1-\delta)L_T^g, & X_T^\pi > \dfrac{L_T^g}{\alpha}, \end{cases} \end{aligned} \quad (2)$$

where $(x)_+ = \max\{x, 0\}$ for a real number x and the liability-to-asset ratio $\alpha \in (0, 1)$. The payoff for the policyholder is equal to the guaranteed amount L_T^g, plus a scaled long option in a call option and a short position in a put. When the terminal portfolio value is