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Optimal hedging with basis risk under mean–variance criterion

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A R T I C L E I N F O

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A B S T R A C T

Basis risk occurs naturally in a number of financial and insurance risk management problems. A notable example is in the context of hedging a derivative where the underlying security is either non-tradable or not sufficiently liquid. Other examples include hedging longevity risk using index-based longevity instrument and hedging crop yields using weather derivatives. These applications give rise to basis risk and it is imperative that such a risk needs to be taken into consideration for the adopted hedging strategy. In this paper, we consider the problem of hedging a European option using another correlated and liquidly traded asset and we investigate an optimal construction of hedging portfolio involving such an asset. The mean–variance criterion is adopted to evaluate the hedging performance, and a subgame Nash equilibrium is used to define the optimal solution. The problem is solved by resorting to a dynamic programming procedure and a change-of-measure technique. A closed-form optimal control process is obtained under a diffusion model setup. The solution we obtain is highly tractable and to the best of our knowledge, this is the first time the analytical solution exists for dynamic hedging of general European options with basis risk under the mean–variance criterion. Examples on hedging European call options are presented to foster the feasibility and importance of our optimal hedging strategy in the presence of basis risk.

1. Introduction

It is well-known in the financial theory that when an option is written on an asset that is tradable, it can be hedged by trading in the underlying asset. What if an option is written on an asset that is either illiquid or even non-tradable? In this case, a common hedging practice is to use another asset that is tradable, highly liquid, and also has the desirable property of being highly correlated to the underlying asset of the option. Because the hedged asset does not perfectly capture the behavior of the underlying asset, there is a mismatch between the risk exposure of the hedged portfolio and the option in question; this gives rise to the so-called basis risk. As shown in Davis (2006), the basis risk could be huge even though both assets have very high correlation. This implies that the basis risk can have a detrimental effect on the hedging performance and hence it needs to be prudently managed.

Basis risk does not just confine to hedging financial derivatives, it exists in many other settings, notably when an index-based security is used for hedging. For example, a pension plan sponsor may choose to hedge the plan's longevity risk by resorting to standard longevity instrument that is traded in the capital market. While such "standard" instrument provides liquidity and transparency, its payoffs are typically determined by mortality indices based on one or more populations. As the longevity experience of the pension plan can deviate significantly from the reference populations, the basis risk, or more specifically, the population basis risk, is said to occur; see also Li and Hardy (2011) and Coughlan et al. (2011). Another example is in the context of managing agricultural risk. In this application, using weather derivatives for hedging agricultural risk could give rise to variable basis risk and spatial basis risk (e.g., Brockett et al., 2005; Woodard and Garcia, 2008). Another situation for which basis risk arises is when a farmer purchases a crop insurance that is based on area yield, instead of individual yield. The area-yield crop insurance, which is known as the Group Risk Plan in the US, is an insurance scheme with indemnity depending on the aggregated county yields. The individual-yield crop insurance, which is known as the Annual Production History Insurance in the US, is another insurance scheme with payoff that is linked to individual farm yields. The discrepancy between yields at the county level and at the individual level gives rise to the basis risk; see for example Skees et al. (1997) and Turvey and Islam (1995).

A typical example in the financial market is that the hedging for an option written on a non-tradable asset is often conducted via trading over one liquidly traded asset which is closely correlated with the non-tradable underlying asset. However, one should

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be very careful to use such a strategy since "close correlation" between the two underlying assets cannot guarantee the hedging performance to be as good as one may desire. Indeed, Davis (2006) showed that the unhedgeable noise, which is attributed to the mismatch between the two assets, may be huge even though the two underlying assets have very high correlation, and the "naive" hedging strategy may be ineffective.

In the existing literature, analytical results on optimal hedging in the presence of basis risk can broadly be classified into two streams. In order to ensure the model's tractability, the first stream of investigation considers hedging general derivatives with basis risk under an exponential utility maximization framework. The pioneering closed-form optimal hedging strategies were obtained by Davis (2006).[1] The basic model of Davis (2006) was subsequently extended by Monoyios (2004) and Musiela and Zariphopoulou (2004) in a few interesting directions including indifference pricing, perturbation expansions, etc. All of these generalizations are restricted to an exponential preference optimization framework. If we were to consider other optimization hedging frameworks such as under a mean–variance criterion, analytical optimal strategies with basis risk have been obtained but only for hedging futures. We classify this line of inquiry as the second stream. The main contribution is attributed to Duffie and Richardson (1991) who obtained the optimal continuous-time futures hedging policy under geometric Brownian motion assumptions. They demonstrated that the optimal hedging strategy can be derived from the normal equations for orthogonal projection in a Hilbert space. Their method, however, is not readily applicable to more general derivatives other than the futures contract. This is because their proposed method depends highly on the specific formulation of the problem and the trivial structure of the payoff function of the futures contract.

Motivated by the above two streams of investigation, this paper attempts to address each of their limitations by studying the dynamic hedging of general European options with basis risk under a mean–variance criterion. Since the seminal work of Markowitz (1952), the mean–variance criterion has been widely applied in finance. A key advantage of the mean–variance criterion over an utility maximization objective is that in practice it is typically challenging to accurately evaluate a hedger's utility function while the mean–variance criterion provides a subjective measure. Furthermore, by comparing to the expected utility approach, MacLean et al. (2011) concluded that, for less volatile financial market, the mean–variance criterion yields a better investment portfolio return.

It is important to emphasize that the optimal portfolio model of Markowitz (1952) is a one-period model. If we are interested in a dynamic portfolio selection strategy, it is important to distinguish optimal strategy that is "pre-commitment" from "time-consistent planning" because of the added possibility of re-optimizing and re-balancing the portfolio at intertemporal times. After a decision maker obtained his/her optimal dynamic strategy at time t_1, the decision maker might find that the adopted strategy from t_1 does not necessarily maximize his/her objective by the time he/she progresses to time t_2, where $t_2 > t_1$. In this situation, the decision maker can either continue to adopt the original plan or to devise a new plan that is "optimal" for him/her at time t_2. Strotz (1955) referred the former strategy as the "precommitment" strategy and the latter as the "consistent planing" strategy. Strotz (1955) also showed that the best investment strategy should be a plan for which the investor will actually follow, e.g., a consistent planing strategy.

[1] Note that the work of Davis (2006) was done in 2000 but it was not formally published until 2006.

The analytical solutions provided by Zhou and Li (2000) and Li and Ng (2000) for, respectively, the continuous-time and multiperiod analogs of Markowitz (1952) are examples of precommitment optimal strategies. To derive the optimal strategies that are time consistent under the mean–variance criterion is considerably more subtle. The complication is driven by the fact that the mean–variance function is not separable so that the Bellman optimality principle cannot be directly applied for deriving an optimal dynamic solution. This problem was not solved until another decade later by Basak and Chabakauri (2010) who provided a novel approach of obtaining a "consistent planing" solution to the portfolio selection problem involving mean–variance objective. They used the total variance formula to derive an extended Hamilton–Jacobi–Bellman (HJB) equation and ingeniously obtain the optimal hedging strategy without directly solving the extended HJB equation as a partial differential equation. Subsequently Björk and Murgoci (2010, 2014) developed a more rigorous theory for general time-inconsistent problems by providing a formal way of defining a "consistent planing" solution using game theoretic approach and providing the verification theorem. In recent years, the time consistent planning strategies have also been widely studied for decision problems in insurance, e.g., Li et al. (2012, 2015a,b), Liang and Song (2015), Wei et al. (2013), Wong et al. (2014), Wu and Zeng (2015), Zeng et al. (2013), Zhao et al. (2016) and Zhou et al. (2016), just to name a few.

In this paper, we aim to establish a "consistent planning" optimal hedging strategy in the sense of Björk and Murgoci (2010). The problem is solved by resorting to a dynamic programming procedure and solving an extended HJB equation using a certain change-of-measure technique. The solution we obtain is tractable and to the best of our knowledge, this is the first time the analytical solution exists for dynamic hedging of general European options with basis risk under the mean–variance criterion. The solution we obtained also reduces to the classical delta hedging strategy when the two involved assets are indistinguishable and the risk aversion coefficient in the mean–variance objective goes to infinity. For plain vanilla call options, the calculation of the optimal strategy requires only a minimum amount of numerical procedure. Examples based on hedging futures and European call options are presented to highlight the importance of our proposed optimal strategy, relative to other commonly adopted hedging strategies that do not take into consideration the basis risk.

The rest of the paper proceeds as follows. The problem formulation is given in Section 2 and the consistent planning equilibrium solution is derived in Section 3. Discussions on some special cases are presented in Section 4. Some numerical examples are provided in Section 5 to highlight our theoretical results. Section 6 concludes the paper. Finally, the Appendix contains some technical proofs and semi closed-form expressions for the equilibrium value functions of both futures and European call options.

2. Formulation of the optimal hedging problem

Let us begin by first introducing the following notations. For a function $F(t, s_1, s_2, x)$, we use F_y to denote its first partial derivative with respect to (w.r.t.) variable y where $y \in \{t, s_1, s_2, x\}$. Analogously, we use F_{yz} to denote its second derivatives w.r.t. variables y and z where $y, z \in \{t, s_1, s_2, x\}$. Note that the function F and its derivatives can be time-dependent processes. In this case, each of the notation will be indexed by an argument t. Similarly, if the arguments s_1, s_2 and x are also processes, then they will be denoted by $S_1(t)$, $S_2(t)$ and $X(t)$, respectively.

Consider a non-arbitrage market with two risky assets $\{S_1(t), t \geq 0\}$ and $\{S_2(t), t \geq 0\}$ as well as a risk-free asset earning at a constant rate of $r > 0$. The price processes of the two risk assets