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Tail subadditivity of distortion risk measures and multivariate tail distortion risk measures

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HIGHLIGHTS

- Study the tail subadditivity for distortion risk measures.
- Obtain sufficient and necessary conditions for the tail subadditivity.
- Propose multivariate tail distortion (MTD) risk measures and give their properties.
- Applications of MTD risk measures in capital allocations for a portfolio of risks.
- Explore the impacts of dependence and extreme tail events on allocations.

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ABSTRACT

In this paper, we extend the concept of tail subadditivity (Belles-Sampera et al., 2014a; Belles-Sampera et al., 2014b) for distortion risk measures and give sufficient and necessary conditions for a distortion risk measure to be tail subadditive. We also introduce the generalized GlueVaR risk measures, which can be used to approach any coherent distortion risk measure. To further illustrate the applications of the tail subadditivity, we propose multivariate tail distortion (MTD) risk measures and generalize the multivariate tail conditional expectation (MTCE) risk measure introduced by Landsman et al. (2016). The properties of multivariate tail distortion risk measures, such as positive homogeneity, translation invariance, monotonicity, and subadditivity, are discussed as well. Moreover, we discuss the applications of the multivariate tail distortion risk measures in capital allocations for a portfolio of risks and explore the impacts of the dependence between risks in a portfolio and extreme tail events of a risk portfolio in capital allocations.

1. Introduction

Risk measures have been extensively used in insurance and finance as a tool of risk management. One of the important functions of risk measures is to determine the required regulatory capitals and to price insurance and reinsurance products. Mathematically, a risk measure is a mapping $\rho : \mathcal{X} \to \mathbb{R} = (-\infty, \infty)$, where $\mathcal{X}$ is a set of loss random variables or risks defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $p \in (0, \infty)$ as the set of all random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite pth moments for $0 < p < \infty$. In particular, L^0 (L^∞) represents the set of all (bounded) random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. For any loss random variable $X \in L^0$, the Value-at-Risk (VaR) of X at a given confidence level $\alpha \in (0, 1)$ is defined as $\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \alpha\} = F_X^{-1}(\alpha)$, which is the left continuous inverse of the distribution $F_X(x) = \mathbb{P}(X \le x) = 1 - S_X(x)$.

When the regulatory capitals of a bank or an insurance company are determined by VaR_α, the solvency probability of the company is at least the confidence level α. However, the risk measure of VaR does not satisfy the subadditivity,[1] which is one property desired by regulators in determining the regulatory capitals. A subadditive risk measure related to VaR is the Tail Value-at-Risk (TVaR). For any loss random variable $X \in L^1$, the TVaR of X at a given confidence level $\alpha \in (0, 1)$ is defined as $\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_q(X)\, dq$.

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[1] A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is said to satisfy the subadditive property if $\rho(X + Y) \le \rho(X) + \rho(Y)$ for all $X, Y \in \mathcal{X}$.

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The risk measure of TVaR is subadditive and satisfies $\mathrm{TVaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(X)$ for any $X \in L^1$. Both VaR and TVaR are the special cases of distortion risk measures. A distortion risk measure $\rho_g : \mathcal{X}_g \to \mathbb{R}$, with a distortion function g, is defined as

$$\rho_g(X) = \int_{-\infty}^{0} [g(S_X(x)) - 1]\mathrm{d}x + \int_{0}^{+\infty} g(S_X(x))\mathrm{d}x. \tag{1.1}$$

Throughout this paper, a function $g : [0, 1] \to [0, 1]$ is said to be a distortion function if g is increasing in $[0, 1]$ with $g(0) = g(0+) = 0$ and $g(1) = g(1-) = 1$. In addition, "increasing" means "non-decreasing" while "decreasing" means "non-increasing". Moreover, for a distortion function g, the risk set $\mathcal{X}_g$ is defined as

$$\mathcal{X}_g = \{X \in L^0 \text{ such that } \rho_g(X) \text{ is finite}\}. \tag{1.2}$$

Note that $g(0+) = 0$ and $g(1-) = 1$ are necessary for the two integrals in (1.1) that are finite when X is an unbounded random variable. The idea behind a distortion risk measure is to distort the survival function of a risk using a distortion function such that the expectation of the risk under the 'distorted measure' can provide more flexible and reasonable measures for the risk. Mathematically, a distortion risk measure ρ_g is the Choquet integral with respect to a set function. That is,

$$\rho_g(X) = \int_{\Omega} X\mathrm{d}\mu_g \overset{\text{def}}{=} \int X\mathrm{d}\mu_g \quad \text{for all } X \in \mathcal{X}_g, \tag{1.3}$$

where μ_g is a monotone set function defined as

$$\mu_g(A) = g(\mathbb{P}(A)) \quad \text{for any } A \in \mathcal{F}. \tag{1.4}$$

A brief review of Choquet integrals and their properties is given in Section 2. Generally speaking, a distortion risk measure of a risk is 'its expectation' over a probability space under a distorted measure.

In terms of Choquet integrals, a distortion risk measure $\rho_g : \mathcal{X}_g \to \mathbb{R}$ is said to be subadditive in $\mathcal{X}_g$, if

$$\int_{\Omega} (X+Y)\,\mathrm{d}\mu_g \leq \int_{\Omega} X\mathrm{d}\mu_g + \int_{\Omega} Y\mathrm{d}\mu_g \tag{1.5}$$

holds for any two random variables $X, Y \in \mathcal{X}_g$. Recently, Belles-Sampera et al. (2014a) have pointed out that subadditivity might be a tough requirement on the determinations of premiums and regulatory capitals. In fact, many practicable risk measures and premium principles do not satisfy subadditivity. Instead of subadditivity, Belles-Sampera et al. (2014a) introduced a weaker concept of tail subadditivity, and investigated the equivalent characterization of this tail subadditivity for a subclass of distortion risk measures, called GlueVaR risk measures. In doing so, they defined a common tail region at a confidence level $\alpha \in (0, 1)$ for any pair of random variables X and Y as

$$\mathfrak{Q}_{\alpha,X,Y} = \{X > F_X^{-1}(\alpha),\ Y > F_Y^{-1}(\alpha),\ X+Y > F_{X+Y}^{-1}(\alpha)\}, \tag{1.6}$$

where F_X and F_X^{-1} are the distribution function and the left continuous inverse of the distribution function of X, respectively. In that paper, a distortion risk measure ρ_g is tail-subadditive for a pair of random variables $X, Y \in \mathcal{X}_g$ at a confidence level $\alpha \in (0, 1)$, if

$$\int_{\mathfrak{Q}_{\alpha,X,Y}} (X+Y)\mathrm{d}\mu_g \leq \int_{\mathfrak{Q}_{\alpha,X,Y}} X\mathrm{d}\mu_g + \int_{\mathfrak{Q}_{\alpha,X,Y}} Y\mathrm{d}\mu_g. \tag{1.7}$$

It should be pointed out that the tail region $\mathfrak{Q}_{\alpha,X,Y}$ is not generally a non-empty set and that there are many other tail regions of different forms that are useful in application/reality. Besides, it will be pointed out later that the equivalent characterization of GlueVaR risk measures presented by Belles-Sampera et al. (2014a) is extended. Interesting applications of the GlueVaR risk measures in insurance, finance, and other fields have been discussed in Belles-Sampera et al. (2014a, b, 2016).

Inspired by this, in this paper, we will generalize the concept of the tail subadditivity for distortion risk measures to any set of the $\mathcal{F}$ and discuss its applications in portfolio risk management. To do so, we first give more general definitions of tail regions and common tail regions.

Definition 1.1. We call a set $\Omega_X \subset \Omega$ a tail region of a random variable X if $\Omega_X \in \mathcal{F}$. Furthermore, we call $\Omega_{\mathbf{X}} = \Omega_{X_1,\ldots,X_n} \subset \Omega$ a common tail region of a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ if $\Omega_{X_1,\ldots,X_n} \in \mathcal{F}$.

We point out that a tail region Ω_X or a common tail region $\Omega_{\mathbf{X}}$ can be chosen as any measurable set in the σ-field $\mathcal{F}$. We call them tail regions since many practical examples of Ω_X or $\Omega_{\mathbf{X}}$ in risk management are based on the tails of random variables or vectors such as those discussed below.

For example, the following tail region of a risk X often arises in solvency risk management:

$$\Omega_{\alpha,X} = \{\omega \in \Omega : X \geq \mathrm{VaR}_\alpha(X)\} = \{X \geq \mathrm{VaR}_\alpha(X)\}, \quad \alpha \in (0, 1). \tag{1.8}$$

In addition, the tail region

$$\Omega_X^e = \{\omega \in \Omega : X \geq \mathbb{E}[X]\} = \{X \geq \mathbb{E}[X]\} \tag{1.9}$$

is an important tail region in premium calculations. Indeed, $\mathrm{VaR}_\alpha(X)$ and $\mathbb{E}[X]$ are respectively the important benchmarks in determining the required solvency capitals and in calculating insurance premiums. Correspondingly, the following two common tail regions

$$\Omega_{\alpha,S_n} = \{S_n \geq \mathrm{VaR}_\alpha(S_n)\}, \tag{1.10}$$

$$\Omega_{S_n}^e = \{S_n \geq \mathbb{E}[S_n]\}, \tag{1.11}$$

are often concerned in portfolio risk management, where $S_n = X_1 + \cdots + X_n$ is the aggregate risk of a portfolio consisting of risks $X_1, \ldots, X_n$. Other interesting common tail regions include

$$\Omega_{\alpha,X_1,\ldots,X_n} = \{X_1 \geq \mathrm{VaR}_\alpha(X_1)\} \cup \cdots \cup \{X_n \geq \mathrm{VaR}_\alpha(X_n)\}, \tag{1.12}$$

$$\Omega_{X_1,\ldots,X_n}^e = \{X_1 \geq \mathbb{E}[X_1]\} \cup \cdots \cup \{X_n \geq \mathbb{E}[X_n]\}, \tag{1.13}$$

$$\Omega_{\alpha,X_1,\ldots,X_n}^* = \{X_1 \geq \mathrm{VaR}_\alpha(X_1)\} \cap \cdots \cap \{X_n \geq \mathrm{VaR}_\alpha(X_n)\}, \tag{1.14}$$

$$\Omega_{X_1,\ldots,X_n}^{e*} = \{X_1 \geq \mathbb{E}[X_1]\} \cap \cdots \cap \{X_n \geq \mathbb{E}[X_n]\}. \tag{1.15}$$

For example, Landsman et al. (2016) have used the common tail region $\Omega_{\alpha,X_1,\ldots,X_n}^*$ to define a multivariate tail conditional expectation (MTCE) risk measure. All the above common tail regions describe the extreme tail events concerned by decision makers in portfolio risk management.

Definition 1.2. For a distortion function g, let $\rho_g : \mathcal{X}_g \to \mathbb{R}$ be the distortion risk measure defined by (1.1) and μ_g be the set function defined by (1.4). The distortion risk measure ρ_g is said to be tail subadditive for a pair of random variables $X, Y \in \mathcal{X}_g$ over a common tail region $\Omega_{X,Y}$, if

$$\int_{\Omega_{X,Y}} (X+Y)\,\mathrm{d}\mu_g \leq \int_{\Omega_{X,Y}} X\mathrm{d}\mu_g + \int_{\Omega_{X,Y}} Y\mathrm{d}\mu_g, \tag{1.16}$$

where the integrals are Choquet integrals over the common tail region $\Omega_{X,Y}$. In addition, $\rho_g : \mathcal{X}_g \to \mathbb{R}$ is said to be tail subadditive in $\mathcal{X}_g$ if (1.16) holds for any pair of $X, Y \in \mathcal{X}_g$ over their common tail region $\Omega_{X,Y}$.

In general, let $\Omega_{\mathbf{X}}$ be a common tail region for a random vector $\mathbf{X} = (X_1, \ldots, X_n)$. The distortion risk measure $\rho_g : \mathcal{X}_g \to \mathbb{R}$ is said