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Optimal consumption–investment strategy under the Vasicek model: HARA utility and Legendre transform☆

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ABSTRACT

This paper studies the optimal consumption–investment strategy with multiple risky assets and stochastic interest rates, in which interest rate is supposed to be driven by the Vasicek model. The objective of the individuals is to seek an optimal consumption–investment strategy to maximize the expected discount utility of intermediate consumption and terminal wealth in the finite horizon. In the utility theory, Hyperbolic Absolute Risk Aversion (HARA) utility consists of CRRA utility, CARA utility and Logarithmic utility as special cases. In addition, HARA utility is seldom studied in continuous-time portfolio selection theory due to its sophisticated expression. In this paper, we choose HARA utility as the risky preference of the individuals. Due to the complexity of the structure of the solution to the original Hamilton–Jacobi–Bellman (HJB) equation, we use Legendre transform to change the original non-linear HJB equation into its linear dual one, whose solution is easy to conjecture in the case of HARA utility. By calculations and deductions, we obtain the closed-form solution to the optimal consumption–investment strategy in a complete market. Moreover, some special cases are also discussed in detail. Finally, a numerical example is given to illustrate our results.

1. Introduction

The consumption–investment problems are the classical portfolio selection problems with consumption behavior in the mathematical finance theory, which were first studied by Merton (1969, 1971). In addition, Merton first used stochastic optimal control theory to study the explicit solution to the optimal portfolio–consumption strategy and laid a solid foundation for the extensive application of stochastic optimal control theory. Since Merton, the consumption–investment problems have been paid great attention by many scholars such that many research results have been achieved. To sum up, these results consist of the following five aspects. (i) The consumption–investment problems with borrowing constraints. One can refer to the works of Fleming and Zariphopoulou (1991) and Vila and Zariphopoulou (1997). Fleming and Zariphopoulou (1991) studied a consumption–investment decision problem for a single agent and used the principle of stochastic dynamic programming to obtain the explicit solution for power utility. In addition, this paper analyzed the optimal portfolio and consumption behavior under the assumption of borrowing rate exceeding the return rate of the risk-free asset. Vila and Zariphopoulou (1997) also used stochastic dynamic programming to study the intertemporal consumption and portfolio choice with borrowing constraint and displayed that how borrowing constraint affected the consumption and portfolio choice. Moreover, Vila and Zariphopoulou (1997) presented a methodology on viscosity solution of the HJB equation, which could be used to analyze a very wide range of consumption/investment problems. (ii) The consumption–investment problems with transaction costs. The interested readers can see the papers of Dumas and Luciano (1991), Shreve and Soner (1994), Dai et al. (2009) and so on. Dumas and Luciano (1991) derived the exact solution to the optimal portfolios with transaction cost. Shreve and Soner (1994) used the method of viscosity solution to obtain the optimal consumption–investment policy with transaction cost and provided a verification theorem.

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Dai et al. (2009) used singular stochastic control theory to deal with the consumption–investment problem with proportional transaction cost and finite time horizon and analyzed the dynamic behavior of the optimal buying and selling strategies. (iii) The consumption–investment problems with stochastic interest rate. Representative works include Fleming and Pang (2004), Munk and Sørensen (2004), etc. Fleming and Pang (2004) focused on the consumption–investment policy with stochastic interest rate on infinite time horizon, and used the sub-supersolution method to prove the existence of the solution of the HJB equation. Munk and Sørensen (2004) characterized the solution to the consumption–investment problem with stochastic interest rates and numerically illustrated an important conclusion that the hedge portfolio is more sensitive to the form of term structure of interest rates than the dynamics of interest rates. (iv) The consumption–investment problems with stochastic volatility. The interested readers can refer to the works of Fleming and Hernandez-hernandez (2003), Chacko and Viceira (2005), Chang et al. (2013), etc. Fleming and Hernandez-hernandez (2003) investigated the optimal consumption problem with stochastic volatility on the infinite horizon and its volatility is an observable economic factor. Chacko and Viceira (2005) supposed the instantaneous volatility of stock price to follow a mean-reverting square-root process and systematically discussed the optimal consumption and portfolio choice of long horizon investors with volatility risks. Chang et al. (2013) assumed the instantaneous volatility of stock price to be related with stock price itself and study the optimal consumption–investment strategy with constant elasticity of variance (CEV) model. (v) The consumption–investment problems with stochastic interest rate and stochastic volatility. Main research results of this aspect include the works of Liu (2007), Noh and Kim (2011), Chang and Rong (2013). The main distinction of these three papers lies in the different framework of problem formulation with stochastic interest rate and stochastic volatility. Liu (2007) assumed that interest rate, the return rate and volatility of stock price were expressed as a function of the stochastic factor, which followed a Markovian diffusion process. Noh and Kim (2011) supposed that the dynamics of interest rate and volatility is generally linearly correlated with the dynamics of stock price. Chang and Rong (2013) assumed that there is no correlationship between the dynamics of stock price and interest rate. These models studied many consumption–investment problems with different market assumptions and different investment environments and greatly extended the works of Merton. But these results were almost studied under the assumption of Constant Relative Risk Aversion (CRRA) utility (i.e. power utility) or logarithmic utility.

It is all well-known that CRRA utility, Constant Absolute Risk Aversion (CARA) utility (i.e. exponential utility) and logarithmic utility are all special cases of HARA utility. As a matter of fact, the investors should choose different utility function according to the different degree of risk preference. Therefore, it is very necessary to investigate the optimal consumption–investment strategy under HARA utility. However, due to the complicated structure of HARA utility, there was little works on the portfolio selection problems with HARA utility in the decade years. Fortunately, we can find that some important results have been achieved in the existing literatures. There are two representative papers in this aspect. One is the work of Grasselli (2003), who presented martingale method to deal with an investment problem with stochastic interest rate and obtained the explicit solutions for HARA utility. Meantime, he also proved the fact that the optimal investment policy under HARA utility converged almost surely to the one under exponential utility and logarithm utility. The other is the work of Jung and Kim (2012), who applied Legendre transform technique to tackle an investment problem under the constant elasticity of variance (CEV) model and achieved the closed-form solution for HARA utility. Apart from these works, Chang and Rong (2014) studied the optimal consumption and investment policy with HARA preference in the constant interest rate environment and obtained the explicit expression by using Legendre transform technique. Chang et al. (2014) used dynamic programming principle along with Legendre transform to investigate the optimal investment problem with random liability and affine interest rate in the HARA framework, and obtained the closed-form solution to the optimal investment strategy. Inspired by these works, we found out that it was feasible to use Legendre transform technique to deal with the consumption–investment problem with stochastic interest rate under HARA utility, which will be discussed in detail in this paper.

In the recent years, Legendre transform technique has been widely used to deal with some complicated portfolio optimization problems, for example, Jonsson and Sircar (2002), Xiao et al. (2007), Gao (2009, 2010) and so on. Precisely speaking, Jonsson and Sircar (2002) presented Legendre transform-dual theory in solving continuous-time portfolio selection problems. Xiao et al. (2007) successfully used Legendre transform-dual theory to obtain the optimal policy for logarithmic utility under a CEV model. Gao (2009) further solved the optimal policy for power utility and exponential utility under the framework of Xiao et al. (2007). Gao (2010) provided Legendre transform-dual rules with four state variables and laid solid foundation for solving more complicated portfolio selection problems.

Considering the fact that interest rate is uncertain in the real-world environments, and more and more scholars began to concern the portfolio selection problems with stochastic interest rates and make the optimal investment strategy obtained more practical. Representative works in this aspect included Korn and Kraft (2001, 2004), Deelstra et al. (2003), Josa-Fombellida and Rincón-Zapatero (2010), Guan and Liang (2014) and so on. Korn and Kraft (2001, 2004) studied the portfolio optimization problems with stochastic interest rate and presented a verification theorem. Moreover, they also discussed some sufficient conditions ensuring the proper application on the principle of stochastic dynamic programming. Deelstra et al. (2003) assumed short rate to be driven by affine interest rate and presented a special stock price model incorporating the effect of interest rate. In addition, they provided the martingale method to study the pension management problem with a minimum guarantee. Josa-Fombellida and Rincón-Zapatero (2010) explored the optimal strategy for pension fund with stochastic interest rate and actuarial liability. Guan and Liang (2014) introduced inflation risk into the pension management problem with stochastic interest rate and obtained the explicit solution to the optimal policy. Nevertheless, these models only studied the optimal investment strategies for CRRA utility or logarithmic utility. Meantime, our analysis suggested that there be many difficulties if we considered consumption behavior in the above literatures and only used stochastic dynamic programming or the martingale method to solve them in the HARA framework. As far as methodology is concerned, most literatures are the univariate optimization problems (for example Grasselli (2003), Jung and Kim (2012)) while the consumption–investment problems are bivariate optimization ones. So our objective function is more complicated, moreover, if we only used the martingale method used by Grasselli (2003), one of the greatest difficulties in dealing with our model lies in the construction of the suitable exponential martingale. If only stochastic dynamic programming was used, we need to directly conjecture the form of the solution to the HJB equation. Due to the complicated nonlinear structure of HARA utility, it is difficult to construct the candidate solution. Compared with the martingale method used by Grasselli (2003), Legendre transform-dual theory is easier to understand and is more convenient to use. By the analysis on the advantages of Legendre transform,