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A note on the convexity of ruin probabilities

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ABSTRACT

Conditions for the convexity of compound geometric tails and compound geometric convolution tails are established. The results are then applied to analyze the convexity of the ruin probability and the Laplace transform of the time to ruin in the classical compound Poisson risk model with and without diffusion. An application to an optimization problem is given.

1. Introduction

The subject of ruin theory has historically attracted much attention in the actuarial literature, and of central importance are the ruin probabilities themselves. Due to their mathematical complexity, various functional aspects of the ruin probabilities have been studied and established. In particular, the ruin probability in the classical compound Poisson (CP) risk model, and more generally the Sparre Andersen risk model, is known to be a compound geometric tail of the form

$$\bar{G}(x) = 1 - G(x) = \sum_{n=1}^{\infty}(1-\phi)\phi^n \bar{F}^{*n}(x), \quad x \geq 0, \tag{1.1}$$

with $0 < \phi < 1$, and $\bar{F}^{*n}(x) = 1 - F^{*n}(x)$ the tail of the distribution of the n-fold convolution of the distribution function (df) $F(x) = 1 - \bar{F}(x)$ with itself, i.e. $\int_0^\infty e^{-sx} F^{*n}(\mathrm{d}x) = \left\{\int_0^\infty e^{-sx} F(\mathrm{d}x)\right\}^n$. See, for example, Chapter VI of Asmussen and Albrecher (2010) for a more detailed discussion. Throughout this paper, we shall assume that F is differentiable with probability density function (pdf) $f(x) = F'(x)$.

Functional properties which carry over from F to G include complete monotonicity, log-convexity, and decreasing (i.e., nonincreasing) failure rate (DFR), among others (e.g., Szekli, 1995, p. 33, and references therein). Compared with these well-known properties, the weaker property of convexity of the compound geometric tail (1.1) has drawn less attention in the ruin theory literature. However, convexity is not only an interesting (albeit challenging) theoretical problem in this context, but it also plays a crucial role in optimization problems involving the ruin probabilities. This includes research problems on the *ruin probability minimization*, a common optimization criterion in the actuarial literature; see, for instance, Young (2004), Promislow and Young (2005), and Bayraktar and Zhang (2015). Note that the minimized ruin probabilities in the aforementioned papers are all convex, except for Bayraktar and Zhang (2015) where the minimized ruin probabilities can nonetheless be converted to a convex function through some transformation. In this paper, we will show that there is a *trap* in the ruin probability minimization problem. In fact, for many jump distributions in classical insurance risk processes, convexity may not hold which causes the standard Hamilton–Jacobian–Bellman (HJB) equation approach in optimization problems to not be directly applicable. Hence, the knowledge of convexity is fundamentally important in an optimization context. As such, this provides the motivation for the present paper where we provide conditions which are either sufficient or necessary for the ruin probability to be convex.

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More generally, we define the tail $\bar{W}(x) = 1 - W(x)$ (of what shall refer to as the *compound geometric convolution*) by

$$\bar{W}(x) = \overline{G * A}(x) = \bar{A}(x) + \int_0^x \bar{G}(x-y)A(\mathrm{d}y), \quad x \geq 0, \tag{1.2}$$

where $A(x) = 1 - \bar{A}(x)$ is an absolutely continuous df with differentiable density $a(\cdot)$ satisfying $A(0) = 0$. The df $A(\cdot)$ is of much interest in a variety of problems in applied probability, including in particular, the classical CP risk process perturbed by a Brownian motion (e.g., Section 9.3 of Willmot and Lin, 2001). As will be demonstrated, this formulation is also sufficient to analyze the non-perturbed case as well (see Section 2.1 for more details). It is well known and easily established that $\bar{W}(x)$ in (1.2) satisfies the defective renewal equation

$$\bar{W}(x) = \phi \int_0^x \bar{W}(x-y)f(y)\mathrm{d}y + \phi\bar{F}(x) + (1-\phi)\bar{A}(x), \tag{1.3}$$

for $x \geq 0$ and that $\bar{W}(x) = \bar{G}(x)/\phi$ in the special case when $A(x) = F(x)$. Thus convexity of $\bar{G}(x)$ may be ascertained as a special case of that of $\bar{W}(x)$. We remark that (1.3) is itself a special case of the general defective renewal equation

$$m(x) = \phi \int_0^x m(x-y)f(y)\,\mathrm{d}y + v(x), \quad x \geq 0,$$

whose solution is given by

$$m(x) = v(x) + \sum_{n=1}^{\infty} \phi^n \int_0^x v(x-y)F^{*n}(\mathrm{d}y), \quad x \geq 0,$$

(see, e.g., Section 3.5 of Resnick, 2013). For related work involving renewal equations, see pp. 811–821 of Hansen and Frenk (1991).

Some general convexity results of $\bar{G}(\cdot)$ and $\bar{W}(\cdot)$ are given in the following lemma which will be used in the later sections.

Lemma 1.1. (a) *The compound geometric tail* $\bar{G}(\cdot)$ *given in* (1.1) *is convex on* $(0, \infty)$ *if* $F(\cdot)$ *is DFR with differentiable density.*
(b) *The compound geometric convolution* $\bar{W}(\cdot)$ *given in* (1.2) *is convex on* $(0, \bar{x})$, *where* $\bar{x} := \sup\{x \geq 0 : q(t) \geq 0$ *for all* $t \in [0, x]\}$ *and*

$$q(x) = -(1-\phi)\left\{a'(x) + \phi a(0)f(x)\right\}, \quad x \geq 0. \tag{1.4}$$

Proof. (a) The result is classical and follows immediately from Proposition 2.1 of Szekli (1986), and Shanthikumar (1988). See also Hipp (1990) for an insurance related discussion.

(b) Assume that the df $A(\cdot)$ has a differentiable density $a(\cdot)$. Differentiation of (1.3) (together with the fact that $\bar{W}(0) = 1$) yields

$$\begin{aligned}\bar{W}'(x) &= \phi \int_0^x \bar{W}'(x-y)f(y)\mathrm{d}y + \phi\bar{W}(0)f(x) \\ &\quad - \phi f(x) - (1-\phi)a(x) \\ &= \phi \int_0^x \bar{W}'(x-y)f(y)\mathrm{d}y - (1-\phi)a(x), \quad x \geq 0.\end{aligned}$$

Differentiating once again, $\bar{W}''(\cdot)$ is shown to satisfy the defective renewal equation

$$\bar{W}''(x) = \phi \int_0^x \bar{W}''(x-y)f(y)\mathrm{d}y + q(x), \quad x \geq 0,$$

where $q(\cdot)$ is defined in (1.4). Clearly, $\bar{W}(\cdot)$ is convex on $(0, \bar{x})$. ■

It is also instructive to note that $\int_0^\infty q(x)\mathrm{d}x = (1-\phi)^2 a(0) \geq 0$, implying that it is not possible that $q(x) < 0$ for all $x \geq 0$.

The rest of the paper is structured as follows. In Section 2, we study the convexity of the ruin probabilities in the CP risk model with and without a diffusion term. Two types of ruin probabilities will be considered, namely infinite-time ruin probabilities and ruin probabilities over an independent exponential time horizon (also referred to as the *Laplace transform of the time to ruin*). The analysis is first performed for the latter, while several conclusions will be made for the former through limiting arguments. An example of non-convex minimized ruin probability is considered in Section 3.

2. Convexity in classical risk models

For completeness, we first introduce the classical CP risk model perturbed by an independent Brownian motion which is defined as

$$X_t = x + ct + \sigma B_t - \sum_{i=1}^{N_t} Y_i, \tag{2.1}$$

where the initial surplus $X_0 = x \geq 0$, the Gaussian coefficient $\sigma \geq 0$, $\{B_t\}_{t\geq 0}$ is a standard Brownian motion, $\{N_t\}_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$, and $\{Y_i\}_{i\geq 1}$ is a sequence of i.i.d. positive rv's with common df $P(\cdot) = 1 - \bar{P}(\cdot)$ and mean $\mu > 0$. We assume that $\{B_t\}_{t\geq 0}$, $\{N_t\}_{t\geq 0}$ and $\{Y_i\}_{i\geq 1}$ are mutually independent.

For the insurance risk process (2.1), let $\tau_0^- = \inf\{t > 0 : X_t < 0\}$. We define the probability of ruin before an independent exponential clock e_δ with mean $1/\delta > 0$ as

$$\begin{aligned}\psi_\delta(x) &= \mathbb{P}\left(\tau_0^- < e_\delta, \tau_0^- < \infty \,|X_0 = x\right) \\ &= \mathbb{E}\left[e^{-\delta\tau_0^-} 1_{\left\{\tau_0^- < \infty\right\}} \,|X_0 = x\right],\end{aligned} \tag{2.2}$$

for $x \geq 0$ where 1_A is the indicator function which takes the value 1 if A is true and 0, otherwise. Note that the infinite-time ruin probability $\psi_0(x) = \mathbb{P}\left(\tau_0^- < \infty \,|X_0 = x\right)$ is a special case of (2.2) with $\delta = 0$. For the infinite-time ruin probability $\psi_0(\cdot)$, it is further assumed that the insurance risk process $\{X_t\}_{t\geq 0}$ has a positive drift, i.e. the positive safety loading condition $c > \lambda\mu$ holds.

2.1. Non-perturbed CP risk model ($\sigma = 0$)

For the non-perturbed CP risk model (i.e., a process $\{X_t\}_{t\geq 0}$ of the form (2.1) with $\sigma = 0$), it is known from e.g., Section 9.2 of Willmot and Lin (2001) that the ruin probability $\psi_\delta(x)$ can be expressed by the compound geometric tail, that is,

$$\psi_\delta(x) = \bar{G}(x) = \frac{\bar{W}(x)}{\phi}, \quad x \geq 0, \tag{2.3}$$

where $\bar{W}(\cdot)$ is as given in (1.2) with

$$\phi = \frac{\lambda}{c} \int_0^\infty e^{-rt}\bar{P}(t)\mathrm{d}t, \tag{2.4}$$

$$F(x) = A(x) = B_r(x) \equiv 1 - \frac{e^{rx}\int_x^\infty e^{-rt}\bar{P}(t)\mathrm{d}t}{\int_0^\infty e^{-rt}\bar{P}(t)\mathrm{d}t}, \quad x \geq 0, \tag{2.5}$$

and $r \in \left[0, \frac{\lambda+\delta}{c}\right)$ is the unique non-negative solution (in s) to $\lambda\widetilde{p}(s) = \lambda + \delta - cs$, where $\widetilde{p}(s) = \int_0^\infty e^{-st}P(\mathrm{d}t)$ is the Laplace–Stieltjes transform of $P(\cdot)$. When $\delta = 0$, note that $r = 0$.

From Theorem 9.2.2 (a) of Willmot and Lin (2001), if the claim size df $P(\cdot)$ is DFR, so is the df $F(\cdot)$ defined in (2.5). Thus, we deduce from part (a) of Lemma 1.1 and (2.3) that $\psi_\delta(\cdot)$ is convex on $(0, \infty)$. More specifically, for the infinite-time ruin probability $\psi_0(\cdot)$, it is easy to see from (2.5) with $r = \delta = 0$ that $F(\cdot)$ is DFR under the weaker condition that the df $P(\cdot)$ has an increasing (i.e. non-decreasing) mean residual lifetime (IMRL) (see, e.g., Section 2.4 of Willmot and Lin, 2001). Once again, from part (a) of Lemma 1.1, $\psi_0(\cdot)$ is convex on $(0, \infty)$ whenever the df $P(\cdot)$ is IMRL.