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Optimal investment and reinsurance for an insurer under Markov-modulated financial market

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ABSTRACT

This study examines optimal investment and reinsurance policies for an insurer with the classical surplus process. It assumes that the financial market is driven by a drifted Brownian motion with coefficients modulated by an external Markov process specified by the solution to a stochastic differential equation. The goal of the insurer is to maximize the expected terminal utility. This paper derives the *Hamilton–Jacobi–Bellman* (HJB) equation associated with the control problem using a dynamic programming method. When the insurer admits an exponential utility function, we prove that there exists a unique and smooth solution to the HJB equation. We derive the explicit optimal investment policy by solving the HJB equation. We can also find that the optimal reinsurance policy optimizes a deterministic function. We also obtain the upper bound for ruin probability in finite time for the insurer when the insurer adopts optimal policies.

1. Introduction

Almost all insurance companies have a reinsurance program. The ultimate goal of the program is to reduce their exposure by passing part of a loss to a reinsurer or a group of reinsurers. With reinsurance, the insurer can issue policies with higher limits, thus being able to take on more risk because some of that risk is now transferred to the reinsurer. Due to this fact and the regulatory requirements, the reinsurance business plays an important role in the operations of an insurance company. Consequently, research on optimal reinsurance policies for insurers has attracted increasing attention from academics and industries. For example, Højgaard and Taksar (1998) studied optimal reinsurance policies for maximizing terminal utility with transaction costs, Schmidli (2001) studied optimal policies for minimizing ruin probability under a diffusion model and the classical risk model, and Taksar and Markussen (2003) studied optimal reinsurance for large insurance portfolios. Besides reinsurance, investment is an increasingly important element in the insurance business that can provide the insurer with potential profit. Thus, optimal investment and reinsurance problems for insurers have drawn great attention in recent years. For example, see Browne (1995), Fleming and Sheu (2000), Gaier et al. (2003), Hipp and Plum (2000), Hipp and Plum (2003), Irgens and Paulsen (2004), Schmidli (2004), Fleming and Pang (2005), Yang and Zhang (2005), Zhang and Siu (2009), Bai and Guo (2008), Fernández et al. (2008), Luo et al. (2008), Azcue and Muler (2009), Zeng and Li (2011), Li et al. (2012), Badaoui and Fernández (2013), Meng et al. (2013) for optimal investment and reinsurance policies under a variety of models and performance functions.

Continuous time frameworks usually assume that the returns from risky assets are stationary (i.e., the coefficients of the dynamics of the returns are constant). Occasionally, closed form solutions to the optimal investment policies are derived under some particular utility function. In reality, the returns from the risky assets might not be stationary. Therefore, it would be of practical relevance and academic importance to consider financial models with non-constant coefficients (see Jean-Peirre et al., 2000; Tankov, 2003; Castañeda-Leyva and Hernández-Hernández, 2005; Fleming and Hernández-Hernández, 2005; Badaoui and Fernández, 2013 for examples). Markov-modulated financial models are one of the most significant models with random coefficients and have been widely used in financial mathematics research (c.f. French et al., 1987). Meanwhile, Markov-modulated risk models contain several very important stochastic volatility models, and can thus be seen as an explanation for many well-known empirical findings, such as volatility smile, volatility clustering, and the heavy-tailed

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nature of return distributions. Thus, for an insurance company, it is worth investigating optimal investment and reinsurance policies in a Markov-modulated financial market. Zeng and Li (2011) studied optimal investment and reinsurance for an insurer in a Helston stochastic volatility model, where the volatility of the financial market is modulated by an external stochastic process. Badaoui and Fernández (2013) studied optimal investment for an insurer in a Markov-modulated model. This paper also focuses on the optimal investment and reinsurance problem for an insurer, assuming that the financial market is Markov modulated and the goal of the insurer is to maximize the expected terminal utility. To some extent, we can view the problem discussed in this paper as a subsequent work of Zeng and Li (2011) by incorporating the classical risk model and considering a more general Markov-modulated financial model, or a subsequent work of Badaoui and Fernández (2013) by introducing reinsurance into the decision. It turns out that when the insurer admits an exponential utility function, the closed form expressions for optimal investment policies are derived and the optimal reinsurance policies can be determined by finding the maximizer of a deterministic function. The upper bound estimation for ruin probability is presented when the insurer adopts the optimal polices.

The rest of this paper is organized as follows. Section 2 presents the model and problem discussed in this paper. A verification theorem of the control problem is presented, which is used to check the results obtained in Section 3. Section 3 focuses on the exponential utility function and gives the closed form expressions for optimal policies when the claims have an exponential distribution. The optimal reinsurance policy can be tracked by solving the maximizer of a deterministic function. Section 4 presents the upper bound estimation for ruin probability when the insurer follows the optimal policies. For reading convenience, some proofs are given in the Appendix.

2. The model, problem and verification theorem

Assume that there are two kinds of assets available for investors: one risky asset and one risk-free asset. One popular framework of risky asset is specified by (c.f. Hipp and Plum, 2003; Yang and Zhang, 2005)

$$dS_t = S_t\left(\mu dt + \sigma dB_{1t}\right), \tag{2.1}$$

and the dynamic of risk-free asset price is

$$dS_t^0 = S_t^0 r dt. \tag{2.2}$$

In this paper, we assume that (2.1) and (2.2) are modulated by an external stochastic process. More precisely, the dynamic of the risky asset in this paper is

$$\begin{cases} dS_t = S_t\big(\mu\left(Z_t\right) dt + \sigma\left(Z_t\right) dB_{1t}\big), \\ S_0 = 1, \end{cases} \tag{2.3}$$

where $\mu(\cdot), \sigma(\cdot)$ are the stochastic investment return rate and volatility of the risky asset, respectively. The dynamic of the external factor is specified by the solution to the following *stochastic differential equation* (SDE)

$$\begin{cases} dZ_t = g\left(Z_t\right) dt + \beta dB_{2t}, \\ Z_0 = z, \end{cases} \tag{2.4}$$

where $B_{1t},\ B_{2t}$ are correlated Brownian motions with coefficient ρ. Our model also contains a risk-free asset specified by

$$dS_t^0 = S_t^0 r\left(Z_t\right) dt, \tag{2.5}$$

where $r(\cdot)$ is the interest rate function. We interpret the process Z_t as the behavior of some economic factor that affects the dynamics of the risky and risk-free assets.

The classical surplus process of an insurer is

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i, \tag{2.6}$$

where $x > 0$ is the initial surplus, c is the positive constant premium income rate, and N_t is the total number of claims up to time t, which is a Poisson process with intensity λ. Y_i denotes the size of the i_{th} claim. Assume that $\{Y_i, i = 1, 2, \ldots\}$ is a sequence of i.i.d positive random variables with finite expectation. Denote the arrival time of the i_{th} claim by T_i. More details about the classical surplus process can be found in Asmussen and Albrecher (2010). We assume that the sequence $\{Y_i,\ i = 1, 2, \ldots,\}$ is independent of $\{N_t,\ t \geq 0\}$ and $\{B_{it}, t \geq 0,\ i = 1, 2\}$.

The insurer has the possibility to invest in the financial market and to take reinsurance. The reinsurer charges the reinsurance premium as adequate compensation for assuming the risk transferred from the insurer. The retention level for the insurer at time t is $b_t \in [0, 1]$, which means that for each claim Y arriving at time t, the part of the claim the insurer pays is $b_t Y$ and that paid by reinsurer is $(1 - b_t)Y$. Accordingly, the insurer has to pay the insurer a premium rate of $\tilde{c}(b_t)$ for this reinsurance. Here, we adopt assumptions about $\tilde{c}(b)$ similar to those in Schmidli (2002).

Since we assume a finite expectation for each individual claim, it is natural to assume that $\tilde{c}(0) < \infty$. We require $\underline{b}$ to prevent that the reinsurance of an insurer's entire claim, and thus an insurer never goes bankrupt for all $x > 0$. This is impossible from practical view. That $\tilde{c}(b)$ is decreasing is natural, otherwise, more reinsurance would be cheaper.

The classical model (2.6) with reinsurance function $c(b)$ is given by

$$R_t^{b,\tilde{c}(b)} = x + \int_0^t \left(c - \tilde{c}\left(b_s\right)\right) ds - \int_0^t b_{s-} d\left(\sum_{i=1}^{N_s} Y_i\right).$$

By adding different forms of $\tilde{c}(b)$, the model discussed here can cover many reinsurance premium principles, for example, see Schmidli (2002) for more details. Let X_t and K_t denote the insurer's wealth and the amount invested in the risky asset at time t, respectively; then, the remaining reserve $X_t - K_t$ is invested in the risk-free asset. Denote the wealth process of the insurer with $X_s = x, Z_s = z$ under investment and reinsurance policies $\{(K_t, b_t), t \in [s, T], 0 \leq s < T\}$ by $X_t^{s,x,z,K,b}$. Once the insurer takes both reinsurance and investment into account, the control system of our problem evolves as

$$\begin{aligned} d\begin{pmatrix} X_t^{s,x,z,K,b} \\ Z_t \end{pmatrix} &= \begin{pmatrix} \left(c - \tilde{c}(b_t) + \big(\mu(Z_t) - r(Z_t)\big)K_t + r(Z_t)X_t^{s,x,z,K,b}\right) \\ g(Z_t) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} K_t\sigma(Z_t) & 0 \\ 0 & \beta \end{pmatrix} d\begin{pmatrix} B_{1t} \\ B_{2t} \end{pmatrix} + \begin{pmatrix} -b_{t-} d\left(\sum_{i=0}^{N_t} Y_i\right) \\ 0 \end{pmatrix} \end{aligned} \tag{2.7}$$

with $X_s = x, Z_s = z, 0 \leq s \leq T$ and with the convention that $\sum_{i=1}^{0} Y_i = 0$.

Definition 2.1. A pair of strategies $(K, b) = \{(K_t, b_t), 0 \leq t \leq T\}$ are *admissible* if they satisfy the following conditions:

(1) The strategies K_t and b_t are predictable with respect to the filtration $\mathcal{F}_t$ given by

$$\mathcal{F}_t = \sigma\left(B_{1s}, B_{2s}, Y_i 1_{[i \leq N_{s-}]}, 0 \leq s \leq t, i \geq 1\right). \tag{2.8}$$