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Characterization of acceptance sets for co-monotone risk measures

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HIGHLIGHTS

- We study acceptance sets of co-monotone, convex risk measures.
- In the case of finite state-spaces we give a complete geometric characterization.
- Applications in low dimensional situations are discussed in detail.

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ABSTRACT

We present a geometric characterization of acceptance sets for monotone, co-monotone and convex risk measures on finite state spaces. Geometrically, such acceptance sets can be represented by convex polygons with edges only on certain hyperplanes. We also provide some lower dimensional examples, and study acceptance sets for value at risk and expected shortfall.

1. Introduction

The oldest risk measure that has been used in finance was variance. It was already introduced into applications by Markowitz (1952). Later, more sophisticated risk measures like value at risk (VaR) have become standard among finance practitioners.

In various applications, however, risk measures are used only indirectly: they define the range of portfolios that have an acceptable risk—the portfolios in the "acceptance set". Mathematically, the key object is therefore in fact not the risk measure, but the acceptance set of portfolios the risk of which is supposed to be tolerable from the perspective of the regulatory authority.

The theory of acceptance sets (and the related concept of capital requirements) has had significant influence on solvency regulations, such as the Basel regimes for banks and Solvency II for insurance companies, and thus on the whole financial sector. There is nowadays a rich theory that axiomatizes "reasonable" acceptance sets. This started with the seminal paper by Artzner et al. (1999) for finite probability spaces, followed by Delbaen (2002) for general probability spaces, and built in part upon a rich literature in actuarial science (Bühlmann, 1970; Gerber, 1979; Goovaerts et al., 1984; Deprez and Gerber, 1985; see also Goovaerts et al., 2010 and the references therein). This notion of "coherent" risk measures then led to the definition of "convex" risk measures (Föllmer and Schied, 2002) and to "entropy coherent" and "entropy convex" risk measures (Laeven and Stadje, 2013). Generalizations to convex acceptance sets and convex capital requirements have been obtained by Föllmer and Schied (2002) and Frittelli and Gianin (2002).

It is interesting to notice that the commonly used VaR fails to satisfy the coherence requirements in these definitions which spurred the interest in trying to find better ways to define acceptance sets.

This leads to the question of how acceptance sets of certain classes of risk measures can be characterized in terms of their intrinsic properties. Besides pure mathematical interest, such characterizations can also be useful to define appropriate acceptance sets for an application that could not easily be defined using risk measures.

In this article we follow this idea and give a characterization of acceptance sets for *co-monotone* and convex risk measures. The

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concept of co-monotonicity has applications in determining provisions for payment obligations in the future, for giving bounds on the price of Asian options and other actuarial and financial problems (Dhaene et al., 2002). Co-monotonicity has also been applied in finding general properties of optimal investment strategies (Rieger, 2011; Hens and Rieger, 2014) that extend classic results by Dybvig (1988).

Our results complement existing results in the literature, in particular, the axiomatization of concave distortions (i.e., law-invariant convex co-monotone risk measures; see Föllmer and Schied, 2011, Chapter 4.6) and of entropy coherent risk measures (i.e., potentially independently (rather than co-monotonically) additive risk measures; see Laeven and Stadje, 2013) in terms of their acceptance sets.

The structure of this article is as follows: Our main characterization result and its derivation will be presented in Section 2. Examples for state spaces with two, three and four states are presented in Section 3. Finally, Section 4 discusses possible extensions and concludes.

2. Acceptance sets of co-monotone risk measures

In this section, we provide a characterization of acceptance sets for monotone, co-monotone and convex risk measures.

Let us start with recalling some basic definitions. For further reading we refer to Denneberg (1994) and Föllmer and Schied (2011).

We always assume that $(\Omega, \mathcal{F})$ is a measurable space, where Ω is a state space and $\mathcal{F}$ its filtration.

Definition 2.1 (*Monotonicity*)**.** A bounded map $\rho\colon L^\infty(\Omega, \mathcal{F})$ is called *monotone* if for all $X, Y \in \Omega$ we have that $X \le Y$ (i.e. $X(\omega) \le Y(\omega)$ for all $\omega \in \Omega$) implies $\rho(X) \ge \rho(Y)$.

Definition 2.2 (*Co-monotonicity of Random Variables*)**.** Let X, Y be random variables on $(\Omega, \mathcal{F})$, then X and Y are *co-monotone* if for all $\omega_1, \omega_2 \in \Omega$:

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \ge 0.$$

Definition 2.3 (*Co-monotone Additivity of Maps*)**.** A bounded map $\rho : L^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is *co-monotone* if for all co-monotone X, Y:

$$\rho(X + Y) = \rho(X) + \rho(Y).$$

Definition 2.4 (*Translation Invariance*)**.** A bounded map $\rho : L^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is *translation invariant* if for all $X \in \Omega$ and $a \in \mathbb{R}$:

$$\rho(X + a) = \rho(X) - a.$$

Definition 2.5 (*Risk Measure*)**.** A bounded, monotone and translation invariant map $\rho : L^\infty(\Omega, \mathcal{F}) \to \mathbb{R}$ is called a *risk measure*.

Assumption 2.6. In the following, we assume $\Omega = \{\omega_1, \dots, \omega_n\}$ with $n \in \mathbb{N}$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_1 = \mathcal{P}(\Omega)$, i.e. a finite state space with no information about the final state initially, but full information at the end.

We will frequently identify random variables on $(\Omega, \mathcal{F})$ with $\mathbb{R}^n$. This can be interpreted as a scenario analysis where a finite number of scenarios that influence the value of the portfolio may occur and ω_i is the value of the portfolio in the case of scenario i.

Our first goal is to find a geometric interpretation of co-monotone functions. This will later enable us to prove a complete classification of acceptance sets for co-monotone risk measures. We start with a number of definitions of geometrical objects that are needed for the subsequent results. In Section 3, we will provide lower dimensional examples illustrating these definitions.

Definition 2.7 (*Dividing Hyperplanes*)**.** Let X be a random variable on $(\Omega, \mathcal{F})$. We define the *dividing hyperplanes* H_{ij} for $i, j \in \{1, \dots, n\}$ by

$$H_{ij} := \{X \in L^\infty(\Omega, \mathcal{F}), X(\omega_i) = X(\omega_j)\}.$$

Identifying random variables on $(\Omega, \mathcal{F})$ with $\mathbb{R}^n$, this translates to

$$H_{ij} = \{X \in \mathbb{R}^n, X_i = X_j\}.$$

We define $H^* := \cap_{i,j} H_{ij} = \{(x, x, \dots, x) | x \in \mathbb{R}\}$.

We note that $H_{ij} = H_{ji}$ and that there are therefore $\binom{n}{2}$ dividing hyperplanes in $\mathbb{R}^n$. We define moreover:

Definition 2.8 (*Halfplanes*)**.** The *positive and negative ij-halfplanes* are:

$$H_{ij}^+(\Omega, \mathcal{F}) := \{X \in L^\infty(\Omega, \mathcal{F}), X(\omega_i) \le X(\omega_j)\}$$
$$H_{ij}^-(\Omega, \mathcal{F}) := \{X \in L^\infty(\Omega, \mathcal{F}), X(\omega_i) \ge X(\omega_j)\}.$$

Again this can be translated into a definition of subsets on $\mathbb{R}^n$:

$$H_{ij}^+ = \{X \in \mathbb{R}^n, X_i \le X_j\}, \qquad H_{ij}^- = \{X \in \mathbb{R}^n, X_i \ge X_j\}.$$

We mention that $H_{ij}^+ = H_{ji}^-$. For the characterization of co-monotonicity the intersections of such halfplanes will be of particular importance. We therefore define:

Definition 2.9 (*Facets*)**.** We call the following subsets of $\mathbb{R}^n$ the *facets* of $\mathbb{R}^n$

$$\mathcal{H} := \left\{ \bigcap_{i,j \in \{1,\dots,n\}, i<j} H_{ij}^{\sigma_{ij}}, \sigma_{ij} \in \{+, -\} \right\}.$$

We note that we do not need to consider the pairs with $i > j$ here, since $H_{ij}^- = H_{ji}^+$. A more tricky point, however, is that some facets are lower dimensional objects, i.e. their interior (the points that have an open neighborhood which is completely contained in the facet) is empty. We can see this in an example: If $H \in \mathcal{H}$ is in the intersection of H_{ij}^+, H_{jk}^+ and H_{ik}^- then $X_i \le X_j \le X_k \le X_i$, thus $X_i = X_j = X_k$. Such facets are just subsets of other facets. We therefore want to exclude them from the definition and call all facets with non-empty interior comonotonicity-subsets or simply c-subsets, for reasons that will become clear later:

Definition 2.10 (*c-subsets*)**.** We define the *co-monotonicity-subsets of* $\mathbb{R}^n$ *(short: c-subsets) as follows*:

$$\{S_k\}_k := \left\{ S_k := \bigcap_{i,j \in \{1,\dots,n\}, i \ne j} H_{ij}^+,\ S_k \subset \mathbb{R}^n \text{ is non-empty} \right\}.$$

How many facets and c-subsets are there in $\mathbb{R}^n$? We have $\frac{n(n-1)}{2}$ pairs (i, j) with $i < j$ in the set $\{1, \dots, n\}$. Thus there are $2^{\frac{n(n-1)}{2}}$ facets in $\mathbb{R}^n$. To calculate the number of c-subsets, we identify the pairs with directed edges in a graph where the vertices are the numbers $\{1, \dots, n\}$ and where the graph is complete (not considering the orientation of its edges).[1] Whenever there is a cycle in the graph, the corresponding facet cannot be a c-subset, since the directed edges lead to a cyclic set of inequalities like in the example above. When there is no cycle, however, the facet is a c-subset. We therefore only need to know how many non-cyclic, complete, directed graphs exist for a given number of vertices.

[1] Compare, e.g., Wallis (2007) for definitions of the graph theoretical terms.