Insurance: Mathematics and Economics 74 (2017) 170–181

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A state dependent reinsurance model

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ARTICLE INFO

Article history:
Received January 2016
Received in revised form February 2017
Accepted 19 March 2017
Available online 31 March 2017

Keywords:
Reinsurance
Time to ruin
Deficit at ruin
Dividend
State dependence

ABSTRACT

We consider the surplus of an insurance company that employs reinsurance. The reinsurer covers part of the claims, but in return it receives a certain part of the income from premiums of the insurance company. In addition, the reinsurer receives some of the dividends that are withdrawn when a certain surplus level b is reached.

A special feature of our model is that both the fraction of the premium that goes to the reinsurer and the fraction of the claims covered by the reinsurer are state-dependent. We focus on five performance measures, viz., time to ruin, deficit at ruin, the dividend withdrawn until ruin, and the amount of money transferred to the reinsurer, respectively covered by the reinsurer.

1. Introduction

An effective way for an insurance company to reduce risk is to buy a reinsurance. According to the reinsurance contract, part of the expenditure burden caused by claims is covered by the reinsurer, and in return the insurance firm transfers part of its income premium to the reinsurer. In addition part of the dividends, that are withdrawn when a certain surplus level b is reached is also transferred to the reinsurer.

The reinsurance may be assumed to be provided instantaneously. In practice, big institutions such as corporations of several big insurance companies, governments or national banks may cover the losses of the insurance firm.

In our model the input is a fluid stream of premiums with general *state-dependent* input rate, and the output is generated by negative *state-dependent* jumps corresponding to the claims that are partially covered by the reinsurer (in a state-dependent way). When the surplus reaches a certain level b (which could be a decision variable) the extra input from premiums is taken as a dividend, so that the surplus is bounded by b. Then the withdrawal of dividend is stopped once the surplus drops below b (at the time of a claim) and so forth.

Let $\tilde{\mathbf{R}} = \{\tilde{R}(t) : t \geq 0\}$ be the *risk-type* process, whose content level is the surplus cash where both the input and the output are state-dependent.

Input: We assume without loss of generality and without any impact on the analysis, that the gross input rate is the constant c, but the net input rate (the dominant factor in the analysis) is a general deterministic function, say $0 < \alpha_R(x) < c$. We modify the process $\tilde{\mathbf{R}}$ as follows: when level b is reached all the extra input from premiums are taken as dividends. Let $\mathbf{R}$ be the modified process. Clearly, $\mathbf{R} \leq b$ and during a dividend period, say I, $\alpha_R(b-)I$ represents the net income from dividend that is taken by the insurance firm, while the part $[c - \alpha_R(b-)]I$ of the dividend is transferred to the reinsurer. Overall, $\alpha_R(x)dx$ for $0 < x \leq b$ is the net amount of infinitesimal input added to the cash of the insurance firm, whenever the state is x.

Output: The net infinitesimal output rate $\beta_R(x)dx$ is a general deterministic function where $0 < \beta_R(x) < 1$; it means that $\beta_R(x)dx$ is the net infinitesimal loss that is subtracted from the content level of the cash, whenever x is downcrossed at moments of claims (negative jumps); the infinitesimal amount $[1-\beta_R(x)]dx$ is covered by the reinsurer.

The policy described above provides a general framework for state dependent claim payments. For a better understanding of how this policy can be implemented consider the following special

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http://dx.doi.org/10.1016/j.insmatheco.2017.03.004
0167-6687/

case: suppose that an arriving claim finds the surplus below a certain threshold level γ, or alternatively, it brings the surplus below that level. Then, the reinsurer covers a certain part of the claim, i.e. $\beta(x) = \beta_0$ for $x < \gamma$. But, whenever the arriving claim does not find or bring the surplus below level γ, the reinsurer pays nothing. In this case $\beta(x) = \beta_0$ for $x < \gamma$ and $\beta(x) = 1$ for $x > \gamma$. In fact, this policy has been introduced in Boxma et al. (2017), but for the discounted model. A natural extension of the latter dichotomous case is to take $\beta(x)$ as a *step function*. That is, let $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\beta(x) = \beta_i$ for $\gamma_{i-1} < x < \gamma_i$. In words, whenever an arriving claim finds the surplus between γ_{i-1} and γ_i or for part of the claim that is between γ_{i-1} and γ_i the reinsurer covers $1 - \beta_i$ of the claim. Then $\beta(x) = \beta_i$ in this strip. In the present study we introduce the general case of arbitrary function β. This arbitrary function includes the special cases of the models mentioned above.

The dynamics described above is a natural procedure of a risk sharing model. However, in order to ease explanations we explain it as a type of reinsurance.

In this study we are interested in analyzing the problem from the point of view of the insurance firm and the reinsurer.

The most interesting five performance measures of this model are (i) the time to ruin, (ii) the deficit at ruin, (iii) the dividend reinsurer withdrawn until ruin, (iv) the amount of money transferred to the until ruin and (v) the total insurance coverage until ruin whose source is the reinsurer. In this paper we shall study the functionals and measures associated with all these five performance measures. An important feature of the paper is the fact that the net premium rate and the net claim sizes are *state-dependent* in a quite general way, giving us considerable modeling flexibility. However, this comes at a price; for example, we only determine the *mean* value of the time to ruin. When more explicit assumptions are being made about the rate functions $\alpha_R(\cdot)$ and $\beta_R(\cdot)$, one might also be able to determine the Laplace transform of the time to ruin (see Boxma et al., 2017).

Related literature

Reinsurance in principle gives rise to multidimensional risk processes. However, despite their obvious relevance, exact analytic studies of multidimensional risk processes are scarce in the insurance literature. An early attempt to assess multivariate risk measures can be found in Sundt (1999), where multivariate Panjer recursions are developed which are then used to compute the distribution of the aggregate claim process, assuming simultaneous claim events and discrete claim sizes. Other approaches are deriving integro-differential equations for the various measures of risk and then iterating these equations to find numerical approximations (Chan et al., 2003; Gong et al., 2012), or computing bounds for the different types of ruin probabilities that can occur in a setting where more than one insurance line is considered (Cai and Li, 2005, 2007). In Badila et al. (2014) a two-dimensional functional equation is taken as a departure point. The authors show how one can find transforms of ruin related performance measures by solving a Riemann–Hilbert type boundary value problem. It is also shown that the boundary value problem has an explicit solution in terms of transforms, if the claim sizes are ordered. In Badila et al. (2015) this is generalized to the case in which the claim amounts are also correlated with the time elapsed since the previous claim arrival.

A special, important case is the setting of proportional reinsurance, which was studied in Avram et al. (2008). There it is assumed that there is a single arrival process, and the claims are proportionally split among two reserves. The two-dimensional exit (ruin) problem then becomes a *one*-dimensional first-passage problem above a piece-wise linear barrier. Badescu et al. (2011) have extended this model by allowing a dedicated arrival stream of claims into only one of the insurance lines. They show that the transform of the time to ruin of at least one of the reserve processes can be derived by applying similar ideas as in Avram et al. (2008).

Bivariate models where one company can transfer its capital to the other have also been considered in the literature. Recently, Avram et al. (2015) proposed a model of an insurance company which splits its premiums between a reinsurance/investment fund and a reserves fund necessary for paying claims. In their setting only the second fund receives claims, and hence all capital transfers are one way: from the first fund to the second. Another example is the capital-exchange agreement in Chapter 4 of Lautscham (2013), or Albrecher and Lautscham (2015) where two insurers pay dividends according to a barrier strategy and the dividends of one insurer are transferred to the other unless the other is also fully capitalized. This work led to systems of integro-differential equations for the expected time of ruin and expected discounted dividends, which are hard to solve even in the case of exponential claims.

In Ivanovs and Boxma (2015) a bivariate risk process is considered with the feature that each insurance company agrees to cover the deficit of the other. Under the assumptions that capital transfers between companies incur a certain proportional cost, and that ruin occurs when neither company can cover the deficit of the other, the survival probability is studied as a function of initial capitals. The bivariate transform of the survival probability is determined, in terms of Wiener–Hopf factors associated with two auxiliary compound Poisson processes. The case of a non-mutual agreement, i.e., reinsurance, is also discussed in Ivanovs and Boxma (2015).

Like the present paper, Boxma et al. (2017) is also devoted to a reinsurance model with an infinitely rich reinsurer, who pays part of the claim when it would bring the surplus below a certain threshold. The focus in that paper is on the discounted case, and on the Gerber–Shiu penalty function.

The features of having a dividend barrier, and of having state-dependent premium rates, appear in quite a few papers in the insurance literature. The following is a far from exhaustive list: Boxma et al. (2010a,b, 2011b), Kyprianou and Loeffen (2010), Lin and Pavlova (2006), Wan (2007) and Zhang et al. (2006).

Finally, we would like to point out that, methodologically, when it comes to studying the density of the surplus capital, this paper bears some relationship to Boxma et al. (2005). The latter paper is concerned with a dam process, and does not consider insurance risk performance measures.

Organization of the paper

In Section 2 we provide some background on the level crossing technique, which is heavily used in the rest of the paper.

The model under consideration is described in Section 3. We there introduce not only the surplus cash model, but also a strongly related dam process (taking $D(t) = b - R(t)$), as well as an other, regenerative, dam process. The five key performance measures mentioned above are studied in Section 5, by relating the surplus cash process to those dam processes. Our results are mostly expressed in the steady-state density of the amount of cash, or of the dam content. That density is determined in Section 4. For the model in full generality, that density is expressed in the form of a Neumann series which is the solution of a Volterra integral equation of the second kind. Under specific assumptions on the claim size distribution and the functions $\alpha_R(\cdot)$ and $\beta_R(\cdot)$, more explicit formulas for the density of the surplus and the five key performance measures can be obtained. In Section 6 we consider the case that the claim arrivals do not follow a Poisson process, but in which the gross negative jump sizes are exponentially distributed. We subsequently consider not only the dam model with $D(t) = b - R(t)$, but we also construct a model that is in a sense dual to that dam model, applying a similar duality as exists between the $M/G/1$ queue and the $G/M/1$ queue (where interarrival and service times are swapped).