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On some multivariate Sarmanov mixed Erlang reinsurance risks: Aggregation and capital allocation

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ABSTRACT

Following some recent works on risk aggregation and capital allocation for mixed Erlang risks joined by Sarmanov's multivariate distribution, in this paper we present some closed-form formulas for the same topic by considering, however, a different kernel function for Sarmanov's distribution, not previously studied in this context. The risk aggregation and capital allocation formulas are derived and numerically illustrated in the general framework of stop-loss reinsurance, and then in the particular case with no stop-loss reinsurance. A discussion of the dependency structure of the considered distribution, based on Pearson's correlation coefficient, is also presented for different kernel functions and illustrated in the bivariate case.

1. Introduction

Modern risk management usually involves complex dependent risk factors. In this respect, several regulations were put in place in order to assess the minimum capital requirement, namely the Economic Capital (EC) that insurance and reinsurance companies are constrained to hold according to their risk exposures. In practice, the EC is evaluated by means of risk measures on the aggregated risk, so that the companies will be covered from unexpected large losses.

For instance, the EC under the Solvency II framework for EU countries focuses on a Value-at-Risk (VaR) approach at a tolerance level of 99.5% of the aggregated risk over a one year period, while in Switzerland, the EC under the Swiss Solvency Test (SST) is based on the Tail-Value-at-Risk (TVaR) approach at a 99% confidence level of the aggregated risk over a one year period. Since the EC quantified in the latter reflects the aggregate capital needed to cover the entire loss of a company, it is also of interest to study how this capital should be allocated among the different risk factors (e.g., lines of business) in the insurance and reinsurance companies, in other words, how much amount of capital each individual risk contributes to the aggregated EC. This allows the risk managers to identify and monitor conveniently their risks. An extensive literature has been developed on capital allocation techniques from which we shall restrict to the TVaR method; see Dhaene et al. (2012), Tasche (2004) and the references therein for more details.

Therefore, the main task of actuaries is to choose an appropriate model for the multivariate risk factors, namely the dependence structure model and the distributions of the marginals. The aim of this contribution is to address risk aggregation and TVaR capital allocation for insurance and reinsurance mixed Erlang risks whose dependency is governed by the Sarmanov distribution with a certain expression of the kernel functions. This study comes along the lines of some recent contributions: Vernic (2017) considered capital allocation based on the TVaR rule for the Sarmanov distribution with exponential marginals; Cossette et al. (2013) used the Farlie–Gumbel–Morgenstern (FGM) distribution to model the dependency between mixed Erlang distributed risks and applied it to capital allocation and risk aggregation; Hashorva and Ratovomirija (2015) and Ratovomirija (2016) presented aggregation and capital allocation in insurance and reinsurance for mixed Erlang distributed risks joined by the Sarmanov distribution

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with a specific kernel function different from the one considered in this study. Note that the choice of the Sarmanov and mixed Erlang distributions is not incidental, these distributions gained a lot of interest in the actuarial literature lately: for the Sarmanov distribution, see e.g., Yang and Hashorva (2013), Hernández-Bastida and Fernández-Sánchez (2012), Abdallah et al. (2016), and Vernic (2016), while for the mixed Erlang distribution we refer to Lee and Lin (2010), Willmot and Lin (2011), Lee and Lin (2012) or Willmot and Woo (2015). One key advantage of the Sarmanov distribution is its flexibility to join different types of marginals and its allowance to obtain exact results. An interesting property of the mixed Erlang distributions is the fact that many risk related quantities, such as TVaR, have an analytical form.

This paper is organized as follows: in Section 2, we present some preliminaries on the Sarmanov distribution, on the TVaR capital allocation problem and on the mixed Erlang distribution, supplemented with several lemmas on this last distribution that will be needed for the proofs of the main results. Section 3 contains the main results on risk aggregation and capital allocation for the stop-loss reinsurance, which are also particularized in the case without stop-loss reinsurance; the main formulas of this section are illustrated with some numerical examples. The paper ends with two appendices: the first one discusses and compares the dependence structure of the bivariate Sarmanov distribution with mixed Erlang marginals and different kernel functions, providing upper and lower bounds for the corresponding Pearson correlation coefficient, while Appendix B contains all the proofs of the theoretical results.

2. Preliminaries

Throughout the paper, by convention, we assume that all the involved quantities exist (i.e., expectations, variances, covariances etc.).

2.1. Multivariate Sarmanov distribution

The Sarmanov distribution caught the interest of many researchers in different fields. It was first introduced by Sarmanov (1966) in the bivariate case, then extended by Lee (1996) to the multivariate case. Its applications in many insurance contexts show its flexible structure when modeling the dependence between multivariate risks given the distribution of the marginals, see the references cited above.

According to Sarmanov (1966), the joint probability density function (pdf) of a bivariate Sarmanov distribution is defined as follows

$$h(x_1, x_2) = f_1(x_1)f_2(x_2)(1 + \alpha_{1,2}\phi_1(x_1)\phi_2(x_2)), \quad x_1, x_2 \in \mathbb{R}, \quad (1)$$

where for $i = 1, 2, f_i$ are the densities of the marginals, ϕ_i are kernel functions assumed to be bounded, non-constant, and $\alpha_{1,2}$ is a real number satisfying the following conditions

$$\mathbb{E}(\phi_i(X_i)) = 0, \quad i = 1, 2, \qquad 1 + \alpha_{1,2}\phi_1(x_1)\phi_2(x_2) \geq 0, \quad \forall\, x_1, x_2 \in \mathbb{R}.$$

Lee (1996) introduced general methods for the choice of ϕ. Yang and Hashorva (2013) considered the case where ϕ depends on some function g, being expressed as follows

$$\phi(x) = g(x) - \mathbb{E}(g(X)), \quad \text{where } \mathbb{E}(g(X)) < \infty.$$

In the context of risk aggregation and capital allocation, Hashorva and Ratovomirija (2015) assumed that $g(x) = e^{-x}$, Vernic (2017) studied the case where the marginals are exponentially distributed, while Cossette et al. (2013) used the FGM distribution with mixed Erlang marginals (the FGM is a special case of the Sarmanov distribution for $g(x) = 2(1 - F(x))$, with F denoting the distribution function of the marginal). Thus, in the sequel, we consider the following kernel function

$$\phi_i(x_i) = f_i(x_i) - \mathbb{E}(f_i(X_i)), \quad (2)$$

where f_i are such that $\mathbb{E}(f_i(X_i)) < \infty, \forall i$. In this case, the range of $\alpha_{1,2}$ is given by

$$\frac{-1}{\max\{\gamma_1\gamma_2, (M_1-\gamma_1)(M_2-\gamma_2)\}} \leqslant \alpha_{1,2} \leqslant \frac{1}{\max\{\gamma_1(M_2-\gamma_2), (M_1-\gamma_1)\gamma_2\}}, \quad (3)$$

where $\gamma_i = \mathbb{E}(f_i(X_i))$ and $M_i = \max_{x\in\mathbb{R}} f_i(x)$, $i = 1, 2$. Moreover, we shall work with a generalization of the above distribution to the multivariate case, see Lee (1996). Hereafter we let $\boldsymbol{X} = (X_1, \ldots, X_n)$ denote an n-dimensional random vector, $\boldsymbol{x} = (x_1, \ldots, x_n)$ (e.g., the observations on $\boldsymbol{X}$) and we let $I_n = 1, \ldots, n$. Therefore, we shall model the dependency between the risks X_i having pdf f_i, $i \in I_n$, via the multivariate Sarmanov distribution having the following pdf

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)\left(1 + \sum_{1\leq j<l\leq n} \alpha_{j,l}\phi_j(x_j)\phi_l(x_l)\right), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad (4)$$

where ϕ_i are the non-constant kernel functions defined in (2) and $\alpha_{j,l}$ are real numbers satisfying the condition

$$1 + \sum_{1\leq j<l\leq n} \alpha_{j,l}\phi_j(x_j)\phi_l(x_l) \geq 0. \quad (5)$$

Remark 2.1. It should be noted that a more general expression of the Sarmanov density for the multivariate case can be written as follows

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)\left(1 + \sum_{l=2}^{n} \sum_{1\leq j_1<j_2<\cdots<j_l\leq n} \alpha_{j_1,\ldots,j_l} \prod_{k=1}^{l} \phi_{j_k}(x_{j_k})\right), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad (6)$$

such that $\mathbb{E}(\phi_i(X_i)) = 0$ and $1 + \sum_{l=2}^{n} \sum_{1\leq j_1<j_2<\cdots<j_l\leq n} \alpha_{j_1,\ldots,j_l} \prod_{k=1}^{l} \phi_{j_k}(x_{j_k}) \geq 0$. However, (6) requires the estimation of all the dependence parameters, which is in general very complex. Thus, it is often assumed that $\alpha_{j_1,\ldots,j_l} = 0$ for $l \geq 3$, see Mari and Kotz (2001). For simplicity, in this paper, we consider the Sarmanov density defined in (4).

2.2. Mixed Erlang distributions

The mixed Erlang distribution has many attractive distributional properties when modeling the claim sizes of an insurance portfolio (see, e.g., Willmot and Lin, 2011) and the dependence between multivariate insurance risks, see Lee and Lin (2012). Actually, during these past few years, modeling the dependence of multivariate mixed Erlang risks raised the interest of many researchers, see also Cossette et al. (2013), Hashorva and Ratovomirija (2015) or Ratovomirija (2016).

In this regard, we define the pdf of a mixed Erlang distribution denoted $ME(\beta, \underline{Q})$ by

$$f(x, \beta, \underline{Q}) = \sum_{k=1}^{\infty} q_k w_k(x, \beta), \quad x \geq 0, \quad (7)$$

where $w_k(x, \beta) = \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!}$ is the pdf of an Erlang distribution with $\beta > 0$ the scale parameter, $k \in \mathbb{N}^*$ the shape parameter and $\underline{Q} = (q_1, q_2, \ldots)$ is a vector of non-negative mixing probabilities such that $\sum_{k=1}^{\infty} q_k = 1$. We denote by W_k the distribution function (df) of the Erlang distribution and by $\overline{W}_k$ its corresponding survival