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Lifetime ruin under ambiguous hazard rate

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ABSTRACT

We determine the optimal robust investment strategy of an individual who targets a given rate of consumption and who seeks to minimize the probability of lifetime ruin when her hazard rate of mortality is ambiguous. By using stochastic control, we characterize the value function as the unique classical solution of an associated Hamilton–Jacobi–Bellman equation, obtain feedback forms for the optimal strategies for investing in the risky asset and distorting the hazard rate, and determine their dependence on various model parameters. We also include numerical examples to illustrate our results, as well as perturbation analysis for small values of the parameter that measures one's level of ambiguity aversion.

1. Introduction

The problem of optimally investing to minimize the probability of running out of money before one dies is an important one, as introduced in Milevsky and Robinson (2000) and analyzed in Young (2004). On a related note, Sid (Browne, 1997) studies a similar survival and growth problem, but for an infinitely-lived agent. The financial and mortality models in Young (2004) and most of the subsequent work in minimizing the so-called *probability of lifetime ruin* have been under the assumption that the models are known; see Bayraktar and Zhang (2015) for references. However, in reality, one does not know the underlying model with certainty; therefore, we want to incorporate model ambiguity in minimizing the probability of lifetime ruin.

Bayraktar and Zhang (2015) allow for model ambiguity in the drift of the risky asset; see their work for relevant references concerning robust control.[1] In this paper, we allow for ambiguity in the hazard rate that affects the individual's mortality. As a rule, individuals do not know the value of their hazard rate, although they might be able to approximate their future life expectancy, whose multiplicative inverse can provide a *reference* hazard rate λ. More specifically, we analyze the robust lifetime ruin problem

$$\inf_{\pi} \sup_{\mathbb{Q}} \left\{ \mathbb{Q}(\tau_0 < \tau_d) - \frac{1}{\varepsilon}\, \mathbb{E}^{\mathbb{Q}} \left[\ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \right\}, \tag{1.1}$$

in which τ_0 and τ_d are the times of ruin and death, respectively, the parameter ε specifies the strength of ambiguity aversion, π runs through possible investment strategies, $\mathbb{Q}$ runs through a set of possible measures representing hazard-rate uncertainty, and $\mathbb{P}$ is the reference measure that incorporates the reference hazard rate λ.[2] Note that the individual's belief about her time of death does

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[1] The mathematical techniques used in that paper are similar to what we use here, but the results here are not trivial and are not comparable to those in Bayraktar and Zhang (2015).

[2] In writing (1.1), we assume that a certain stochastic integral has zero $\mathbb{Q}$-expectation; see Eqs. (2.2) and (2.3).

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not follow a true exponential distribution, but something close to it, as measured by entropic distance. The effect of our model ambiguity is illustrated in Section 6. For example, to better protect herself from the risk of over-estimating the hazard rate, the agent should invest more aggressively when she is poor.

One can think of the optimization problem in (1.1) as minimizing the probability of lifetime ruin, as in Young (2004), with a penalty for ambiguity concerning the hazard rate. Here, we distinguish between ambiguity and stochasticity in the hazard rate. In the latter case, the random hazard rate has a known distribution, whereas in the former case, the distribution is unknown. For a literature review of stochastic mortality models, see, for example, Dahl (2004) or Cairns et al. (2008). For tractability, we work with a constant reference hazard rate. In future research, it would be interesting to incorporate ambiguity aversion to time-dependent or even stochastic mortality models. We also plan to apply a similar penalty to other goal-seeking problems, such as maximizing the probability of reaching a bequest goal, as in Bayraktar and Young (2015).

When $\varepsilon \to 0$ and $\varepsilon \to \infty$, we obtain explicit formulas for the value function and corresponding optimal controls. For $0 < \varepsilon < \infty$, we characterize the value function as the unique classical solution of an associated Hamilton–Jacobi–Bellman (HJB) equation satisfying two boundary conditions, and give feedback forms for the optimal strategies of investing in the risky asset and distorting the hazard rate. For our analysis, we cite much of the work of Bayraktar and Zhang (2015); therefore, we focus on the properties of the resulting robust value function.

The rest of the paper is organized as follows. In Section 2, we define the Black–Scholes financial market in which the individual invests and consumes. More importantly, in that section, we develop the set $\mathcal{Q}$ of probability measures $\mathbb{Q}$ that model uncertainty in the individual's hazard rate; then, at the end, we define the corresponding robust value function, a penalized minimum probability of lifetime ruin. In Section 3, we state our main theorem, Theorem 3.1, which characterizes the robust value function as the unique solution of the corresponding HJB equation with appropriate boundary values. Theorem 3.1 also presents, in feedback form, the optimal controls for investing in the risky asset and for distorting the hazard rate. Then, in Section 4, we outline a proof of Theorem 3.1 and provide details, as needed. This section also includes the interesting result that both the robust value function and optimal investment in the risky asset increase as one's ambiguity aversion increases. We discuss this property and prove other properties of the robust value function and optimal controls in Section 5. Finally, in Section 6, we provide some numerical examples that illustrate our analytical results, and we asymptotically expand the robust value function for small values of ε, the parameter that represents the investor's level of ambiguity aversion.

2. Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space supporting a Brownian motion $B = (B_t)_{t\geq 0}$ and an exponential random variable τ_d that is independent of $\mathbb{F}$ and has rate λ. τ_d models the death time of the individual. In this context, we consider λ to be the *reference* hazard rate. The value of λ is uncertain, and we model the individual's ambiguity about it.

Let $D_t := \mathbf{1}_{\{\tau_d \leq t\}}$ be the death indicator process; $D = (D_t)_{t\geq 0}$ jumps from 0 to 1 when the individual dies. Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the progressive enlargement of the filtration $\mathbb{F}$ to include information generated by D, specifically, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(D_u : 0 \leq u \leq t)$. Assume $\mathbb{F}$ and $\mathbb{G}$ have been augmented to satisfy the usual condition of completeness and right continuity. Under the reference measure $\mathbb{P}$, D has jump rate $\lambda \mathbf{1}_{\{D_t=0\}}$, and $M_t^{\mathbb{P}} := D_t - \int_0^t \lambda \mathbf{1}_{\{D_u=0\}} du$ forms a $(\mathbb{P}, \mathbb{G})$-martingale.

The financial market consists of a riskless bank account with interest rate $r > 0$ and a risky asset whose price process $S = (S_t)_{t\geq 0}$ follows a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

with $S_0 > 0$, $\mu > r$, and $\sigma > 0$. Let π_t be the dollar amount that the individual invests in the risky asset at time t. We say $(\pi_t)_{t\geq 0}$ is admissible if it is $\mathbb{F}$-progressively measurable and satisfies $\int_0^t \pi_u^2 \, du < \infty \mathbb{P}$-a.s. for all $t \geq 0$. Denote by $\mathscr{A}$ the set of all admissible investment strategies. Apart from investment, the individual also consumes at a constant rate $c > 0$. Her wealth W_t° has the following pre-death dynamics:

$$dW_t^\circ = \left(rW_t^\circ + (\mu - r)\pi_t - c\right) dt + \sigma \pi_t dB_t, \quad W_0^\circ = w. \tag{2.1}$$

Define the time of ruin $\tau_0 := \inf\{t \geq 0 : W_t^\circ \leq 0\}$ to be the first time the individual's wealth falls to or below zero. The individual aims to minimize the probability that ruin occurs before her death, that is, $\tau_0 < \tau_d$, but she is concerned that the hazard rate might be misspecified. So, instead of optimizing under the reference measure $\mathbb{P}$, she considers a set $\mathcal{Q}$ of candidate measures, and penalizes a given measure's deviation from $\mathbb{P}$.

We assume the individual is only ambiguous about the hazard rate, not the financial market (drift uncertainty is studied in Bayraktar and Zhang (2015)). However, under alternative measures, we allow the hazard rate to depend on the information in the financial market. More precisely, a probability measure $\mathbb{Q}$ is in $\mathcal{Q}$ if $d\mathbb{Q}/d\mathbb{P} = L_\infty = \lim_{t\to\infty} L_t$, in which

$$L_t := \mathcal{E}\left(\int_0^\cdot (\vartheta_{u-} - 1)\, dM_u^{\mathbb{P}}\right)_t = \exp\left(\int_0^t \ln(\vartheta_{u-})\, dD_u - \int_0^t \lambda \mathbf{1}_{\{D_u=0\}}(\vartheta_u - 1) du\right),$$

for some $\mathbb{F}$-progressively measurable, positive process $\vartheta = (\vartheta_t)_{t\geq 0}$. In other words, L_t is the solution of

$$dL_t = L_{t-}(\vartheta_{t-} - 1)\, dM_t^{\mathbb{P}}, \quad L_0 = 1.$$

(See page 59 of Jacod and Shiryaev, 2003 for the Doléans-Dade exponential formula $\mathcal{E}$.) Observe that $L_t = L_{t\wedge\tau_d}$; thus, L_∞ exists. To ensure that $\mathbb{E}^{\mathbb{P}}[L_\infty] = 1$, we only consider those ϑs under which $L = (L_t)_{t\geq 0}$ a uniformly integrable $(\mathbb{P}, \mathbb{G})$-martingale. This will be the case when ϑ is bounded away from zero and infinity.[3] From Bielecki et al. (2009, Theorem 3.4.1), we know B is a $(\mathbb{Q}, \mathbb{G})$-Brownian motion and the $\mathbb{Q}$-intensity of D is $\lambda \vartheta_t \mathbf{1}_{\{D_t=0\}}$, that is, $M_t^{\mathbb{Q}} := D_t - \int_0^t \lambda \vartheta_u \mathbf{1}_{\{D_u=0\}}\, du$ forms a $(\mathbb{Q}, \mathbb{G})$-martingale.

Use $M^{\mathbb{Q}}$ to rewrite L_t as

$$L_t = \exp\left(\int_0^t \ln(\vartheta_{u-})\, dM_u^{\mathbb{Q}} + \int_0^t \lambda \mathbf{1}_{\{D_u=0\}}(\vartheta_u \ln \vartheta_u - \vartheta_u + 1)\, du\right).$$

The relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$\mathbb{E}^{\mathbb{Q}}[\ln L_\infty] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \ln(\vartheta_{u-})\, dM_u^{\mathbb{Q}} + \int_0^\infty \lambda \mathbf{1}_{\{D_u=0\}}(\vartheta_u \ln \vartheta_u - \vartheta_u + 1)\, du\right]. \tag{2.2}$$

By assuming the stochastic integral vanishes upon taking $\mathbb{Q}$-expectation, we have

$$\mathbb{E}^{\mathbb{Q}}[\ln L_\infty] = \mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty \lambda \mathbf{1}_{\{D_u=0\}}(\vartheta_u \ln \vartheta_u - \vartheta_u + 1)\, du\right]. \tag{2.3}$$

[3] Indeed, if ϑ is bounded away from zero and infinity, then $\exists \delta > 0$ such that $\sup_\tau \mathbb{E}^{\mathbb{P}}[L_\tau^{1+\delta}] < \infty$, in which the supremum is taken over all finite $\mathbb{G}$-stopping times, which implies L is of class (D), thus, is a uniform integrable martingale.