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Preserving the Rothschild–Stiglitz type of increasing risk with background risk

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ABSTRACT

Background risk refers to a risk that is exogenous and is not subject to transformations by a decision-maker. In this paper, we extend the definition of the Rothschild–Stiglitz type of increasing risk to a background risk framework. We theoretically investigate a more general definition of increase in risk in the presence of background risk. The results suggest that an extended concept of expectation dependence plays a vital role.

1. Introduction

Comparisons of risk have been of particular interest to researchers in economics and finance. Many approaches to address this problem have been developed over the past few decades. One of the essential works in this field is the seminal work of Rothschild and Stiglitz (hereafter R–S) (1970). In their work, R–S propose the following two equivalent definitions of increasing risk:

(a) $\tilde{x}_1$ is riskier than $\tilde{x}_2$ if $\tilde{x}_1$ is equal to $\tilde{x}_2$ plus a "noise":

$$\tilde{x}_1 = \tilde{x}_2 + \tilde{\varepsilon}, \tag{1}$$

where $E(\tilde{\varepsilon}|\tilde{x}_2 = x_2) = 0$ for all x_2.

(b) $\tilde{x}_1$ is riskier than $\tilde{x}_2$ if both have the same mean and $Eu(\tilde{x}_1) \geq Eu(\tilde{x}_2)$ for all concave u.

The question studied by Rothschild and Stiglitz (1970) was: What is the more general definition of increasing risk that is compatible with an expected utility function? The equivalence of the above definitions allows the concept of increasing risk to be applied in both expected utility and non-expected utility settings. Subsequently, many studies have used the R–S definition of increasing risk in economic and finance applications. The literature was summarized and presented by Meyer (2014) and Dionne and Harrington (2014).

In many situations, a decision-maker faces two risks at the same time: one is exogenous and is not subject to transformation by the decision-maker, while the other is endogenous and can be controlled. We call the exogenous risk the background risk. Over past decades, many studies of background risk have been conducted. For example, insurance and optimal portfolio models that integrate background risk have been put forth to solve various puzzles in economics and finance.[1]

In this paper, we generalize the R–S definition to a background risk setting. We first extend the concept of expectation dependence proposed by Wright (1987), and introduce a definition to compare expectation dependence between random variables. We then

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[1] See Tsetlin and Winkler (2005), Hong et al. (2011), Li (2011), and Dionne and Li (2014) for recent contributions.

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extend the definition of the R–S type of increasing risk to a background risk setting. We investigate the conditions under which our definition of increasing risk is compatible with a von Neumann Morgenstern utility function.

The paper proceeds as follows. Section 2 first reviews the concept of expectation dependence and then introduces several new concepts. Section 3 investigates the definition of the R–S type of increasing risk in the presence of background risk. Section 4 concludes the paper.

2. Some concepts

Suppose that $\tilde{x}_1 \times \tilde{x}_2 \times \tilde{y}_1 \times \tilde{y}_2 \in [\underline{x}_1, \bar{x}_1] \times [\underline{x}_2, \bar{x}_2] \times [\underline{y}_1, \bar{y}_1] \times [\underline{y}_2, \bar{y}_2]$ is a 4-dimensional random vector. Wright (1987) proposes the following concept.

Definition 2.1 (*Wright, 1987*)**.** If

$$ED(\tilde{x}_1|\tilde{y}_1 \le y_1) = [E\tilde{x}_1 - E(\tilde{x}_1|\tilde{y}_1 \le y_1)] \ge 0 \quad \text{for all } y_1, \tag{2}$$

then $\tilde{x}_1$ is positive expectation dependent on $\tilde{y}_1$. The family of all distributions satisfying (2) is denoted by $(\tilde{x}_1, \tilde{y}_1) \in ED(\tilde{x}_1|\tilde{y}_1 \le y_1)$.

Wright (1987) interprets $(\tilde{x}_1, \tilde{y}_1) \in ED(\tilde{x}_1|\tilde{y}_1 \le y_1)$ as follows. When we know that $\tilde{y}_1$ is truncated from above ($\tilde{y}_1 \le y_1$), the expectation of $\tilde{x}_1$ decreases. We then have the following definition.

Definition 2.2. If

$$\begin{aligned} ED(\tilde{x}_1|\tilde{y}_1 \le y_1, \tilde{y}_2 \le y_2) &= [E\tilde{x}_1 - E(\tilde{x}_1|\tilde{y}_1 \le y_1, \tilde{y}_2 \le y_2)] \\ &\ge 0 \quad \text{for all } (y_1, y_2), \end{aligned} \tag{3}$$

then $\tilde{x}_1$ is positive expectation dependent on $(\tilde{y}_1, \tilde{y}_2)$. The family of all distributions satisfying (3) is denoted by $(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2) \in ED(\tilde{x}_1|\tilde{y}_1 \le y_1, \tilde{y}_2 \le y_2)$.

We can interpret $(\tilde{x}_1, \tilde{y}_1, \tilde{y}_2) \in ED(\tilde{x}_1|y_1, y_2)$ as follows. When we know that $\tilde{y}_1$ and $\tilde{y}_2$ are truncated from above ($\tilde{y}_1 \le y_1$ and $\tilde{y}_2 \le y_2$), the expectation of $\tilde{x}_1$ decreases. As $ED(\tilde{x}_1|\tilde{y}_1 \le y_1, \bar{y}_2) = ED(\tilde{x}_1|y_1)$, Definition 2.2 generalizes Definition 2.1.

We introduce the following definition to compare the dependence between random variables.

Definition 2.3. If

$$\begin{aligned} &ED(\tilde{x}_1|\tilde{y}_1 \le y_1, \tilde{y}_2 \le y_2) \ge ED(\tilde{x}_2|\tilde{y}_1 \le y_1, \tilde{y}_2 \le y_2) \\ &\text{for all } (y_1, y_2), \end{aligned} \tag{4}$$

then $\tilde{x}_1$ is more expectation dependent than $\tilde{x}_2$ on $(\tilde{y}_1, \tilde{y}_2)$. The family of all distributions satisfying (4) is denoted by $(\tilde{x}_1; \tilde{y}_1, \tilde{y}_2) \succeq_{ED} (\tilde{x}_2; \tilde{y}_1, \tilde{y}_2)$.

We recall another measurement of dependence: "more concordance".

Definition 2.4 (*Tchen, 1980*; *Epstein and Tanny, 1980*)**.** $(\tilde{x}_1, \tilde{y}_1)$ is more concordant than $(\tilde{x}_2, \tilde{y}_2)$ if $F_{X_1}(x) = F_{X_2}(x)$, $F_{Y_1}(y) = F_{Y_2}(y)$ and $F_{X_1Y_1}(x, y) \ge F_{X_2Y_2}(x, y)$ for all (x, y).

A simple comparison between the above two dependence concepts shows that the marginal distributions of $(\tilde{x}, \tilde{y})$ in "more concordance" must be the same, but this is not the case in "more expectation dependence".

3. Model

Denote $u(x, y)$ as the utility function, and let $u_1(x, y)$ denote $\frac{\partial u}{\partial x}$ and $u_2(x, y)$ denote $\frac{\partial u}{\partial y}$. We follow the same subscript convention for the derivatives $u_{11}(x, y)$, $u_{12}(x, y)$ and so on, and assume that the partial derivatives required for any definition all exist and are continuous and bounded.

We propose an extension of the definition of increasing risk proposed by Rothschild and Stiglitz to the presence of a background risk.

Definition 3.1. $(\tilde{x}_1; \tilde{y}_1)$ is the R–S type of increasing risk with respect to $(\tilde{x}_2; \tilde{y}_1)$ if the following condition is satisfied:

$$Eu(\tilde{x}_1, \tilde{y}_1) \le Eu(\tilde{x}_2, \tilde{y}_1) \quad \text{for } E\tilde{x}_1 = E\tilde{x}_2. \tag{5}$$

This definition states that, although $E\tilde{x}_1 = E\tilde{x}_2$, a change in risk from $(\tilde{x}_2, \tilde{y}_1)$ to $(\tilde{x}_1, \tilde{y}_1)$ makes the agent worse off. The following proposition shows the conditions under which (5) is satisfied.

Proposition 3.2. *If* (i) $(\tilde{x}_1; \tilde{x}_2, \tilde{y}_1) \succeq_{ED} (\tilde{x}_2; \tilde{x}_2, \tilde{y}_1)$*;* (ii) $u_{11} \le 0$, $u_{12} \le 0$ *and* $u_{112} \ge 0$*, then* (5) *holds.*

Proof. See Appendix. □

Proposition 3.2 states that if $\tilde{x}_1$ is more expectation dependent than $\tilde{x}_2$ on $(\tilde{x}_2, \tilde{y}_1)$, then an agent who is risk averse in x ($u_{11} \le 0$), correlation averse ($u_{12} \le 0$) and cross-prudent ($u_{112} \ge 0$) dislikes a change in risk from $(\tilde{x}_2, \tilde{y}_1)$ to $(\tilde{x}_1, \tilde{y}_1)$.

We now introduce a definition of R–S increasing risk for two risks.

Definition 3.3. $(\tilde{x}_1; \tilde{y}_1)$ is the R–S type of increasing risk with respect to $(\tilde{x}_2, \tilde{y}_2)$ if the following condition is satisfied:

$$\begin{aligned} &Eu(\tilde{x}_1, \tilde{y}_1) \le Eu(\tilde{x}_2, \tilde{y}_2) \quad \text{for } E\tilde{x}_1 = E\tilde{x}_2 \\ &\text{and} \quad E\tilde{y}_1 = E\tilde{y}_2. \end{aligned} \tag{6}$$

This definition states that a mean-preserving spread from $(\tilde{x}_2, \tilde{y}_2)$ to $(\tilde{x}_1, \tilde{y}_1)$ makes the agent worse off. Proposition 3.4 provides the sufficient conditions for (6).

Proposition 3.4. *If* (i) $(\tilde{x}_1; \tilde{x}_2, \tilde{y}_1) \succeq_{ED} (\tilde{x}_2; \tilde{x}_2, \tilde{y}_1)$ *and* $(\tilde{y}_1; \tilde{x}_2, \tilde{y}_2) \succeq_{ED} (\tilde{y}_2; \tilde{x}_2, \tilde{y}_2)$*;* (ii) $u_{11} \le 0$, $u_{22} \le 0$, $u_{12} \le 0$, $u_{112} \ge 0$ *and* $u_{122} \ge 0$*, then* (6) *holds.*

Proof. See Appendix. □

This proposition states that if $\tilde{x}_1$ is more expectation dependent than $\tilde{x}_2$ on $(\tilde{x}_2, \tilde{y}_1)$ and $\tilde{y}_1$ is more expectation dependent than $\tilde{y}_2$ on $(\tilde{x}_2, \tilde{y}_2)$, then an agent who is risk averse in x and y ($u_{11} \le 0$ and $u_{22} \le 0$), correlation averse ($u_{12} \le 0$) and cross-prudent ($u_{112} \ge 0$ and $u_{122} \ge 0$) dislikes a mean-preserving change in risk from $(\tilde{x}_2, \tilde{y}_2)$ to $(\tilde{x}_1, \tilde{y}_1)$.

4. Conclusion

The R–S definition of increasing risk is an important concept in many economics and finance studies. The main contribution of this paper is to extend the R–S definition to a background risk framework. As a number of problems in economics, finance, insurance, and generally in decision making under uncertainty fall within a background risk framework, our results may be useful for such applications.

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