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Borch's theorem, equal margins, and efficient allocation

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ABSTRACT

The economic concept of margin guides or justifies the sharing of risks and resources. Broadly, by *Borch's theorem*, competing claimants, ends or users ought see *equal margins* along any *efficient allocation*.

However helpful this maxim, its application is often hampered, and occasionally misguided, by concerns with the differentiability of objectives—or with the interiority of solutions. Circumventing such concerns, this paper introduces a quite applicable, generalized notion, called *essential margin*.

Presuming transferable or quasi-linear utility, the coincidence of such margins supports efficiency, competitive equilibria, and core solutions. The said coincidence also defines deductibles and prioritized claims, seen in finance and insurance.

1. Introduction

When well defined, the concept of *margin* is fundamental in economics. To wit, for efficient allocation, *margins ought coincide across alternative ends and users*. Otherwise, scarce resources should be shifted from low valuation (or from inferior yield) to higher.

Traditional use of this good maxim requires though, comparisons of differentials or gradients. For that reason, several questions come straight up: What happens if "gradients" are not unique—or, no less important, if a best choice be at the boundary?In such cases, which margins are essential? And how might these coincide?

While addressing these questions, this paper maintains and refines the said maxim, often referred to as *Borch's theorem* of insurance (Borch, 1962; Cheung et al., 2014), and extends its domain of application. Covered below are manifold instances by one umbrella. Presuming transferable or quasi-linear utility, the paper opens blitz avenues to efficiency, competitive equilibria and core outcomes. It also illuminates the priority rules which affect many insurance policies or financial securities (Arrow, 1963).

On a technical note, the approach, chosen here, dispenses with classical differentiability, and it subscribes to convex analysis. Yet, in some settings, the role of convexity is also played down; it imports chiefly at the aggregate level. Instrumental for the analysis are maximal convolutions of individual criteria. The optic is geometric in that gradients, if any, stand orthogonally on budget sets, utility curves or technologies. However – absent interiority, or smoothness, or finite dimension – orthogonality is better described by what is called *normal cones*.

On a didactical note, the paper illustrates the unifying simplicity of generalized gradients. It invokes few requisites and is meant to be accessible for diverse readers. Proofs are direct and simple. Except for reasons of exposition, vector spaces can be general, and topological arguments recede into the background.

Section 2 states the allocation problem and some motivating examples. Section 3 introduces what qualifies as *essential margins*, and it brings out that their coincidence is paramount for efficiency. Section 4 inquires about the role of coinciding margins for competitive equilibria and core outcomes. Section 5 considers mutual insurance and the sequential structure of coverage. Section 6 concludes.

Notations and preliminaries. Henceforth $\mathbb{X}$ denotes a real vector space. Its *dual space* $\mathbb{X}^*$ consists of all linear functionals from $\mathbb{X}$

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into $\mathbb{R}$. Continuity of such functionals is an important but separate issue, not addressed here. Anyway, at a point x, which belongs to the *effective domain*

$$dom f := f^{-1}(\mathbb{R}) =: \{x \in \mathbb{X} : f(x) \in \mathbb{R}\} \tag{1}$$

of a function $f : \mathbb{X} \to \mathbb{R} \cup \{\pm\infty\}$, declare $x^* \in \mathbb{X}^*$ a *generalized gradient* (or *supergradient*) or *essential margin*, as signaled by writing $x^* \in \partial f(x)$, iff

$$f(\chi) \le f(x) + x^*(\chi - x) \quad \text{for all } \chi \in \mathbb{X}. \tag{2}$$

In particular, an *extended indicator* $\iota_X : \mathbb{X} \to \{-\infty, 0\}$ of a subset $X \subseteq \mathbb{X}$, defined by

$$\iota_X(x) =: \iota(X, x) = 0 \Longleftrightarrow x \in X,$$

generates, by way of generalized differentiation, an outward *normal cone*[1]

$$N(X, x) := \left\{x^* \in \mathbb{X}^* : x^*(\chi - x) \le 0 \forall \chi \in X\right\} = -\partial_X(x) \tag{3}$$

at $x \in X$. Note that $N(X, x) = \{0\}$ at each member of the *algebraic interior*

$$aintX := \{x : x \text{ is relatively interior to } X \cap L \text{ for every line } L \text{ through } x\}.$$

2. The efficiency problem

Accommodated henceforth is a fixed, finite ensemble $I = \{1, \ldots, |I|\}$ of economic agents, $|I| \ge 2$. Member $i \in I$ enjoys *transferable* or *quasi-linear utility* $\mathcal{U}_i(x_i) \in \mathbb{R} \cup \{\pm\infty\}$, in the nature of pecuniary payoff derived from "input" x_i. The latter item – being a commodity bundle, contingent claim, or risk – is codified as a vector in some real linear space $\mathbb{X}$, the same space for all agents. Besides the implicit restriction that x_i belongs to the *effective domain* (1)

$$dom\mathcal{U}_i = \{x_i \in \mathbb{X} : \mathcal{U}_i(x_i) \in \mathbb{R}\} =: X_i,$$

there is also the coupling constraint

$$\sum_{i \in I} x_i = x_I, \tag{4}$$

$x_I \in \mathbb{X}$ being a prescribed aggregate "endowment" which the agents should share. A profile $i \in I \mapsto x_i \in \mathbb{X}$ that satisfies (4) is called an *allocation*, written (x_i). It is *feasible* when moreover, $x_i \in X_i \ \forall i$, and (Pareto) *efficient* if it realizes the value

$$\mathcal{U}_I(x_I) := \sup\left\{\sum_{i \in I} \mathcal{U}_i(x_i) : \sum_{i \in I} x_i = x_I\right\}. \tag{5}$$

By standing assumption, a feasible allocation exists, hence $\mathcal{U}_I(x_I) > -\infty$.

My concerns are with characterization of efficiency. Borch's theorem (Borch, 1962), dealing with the case of expected utility $\mathcal{U}_i = Eu_i$, indicates that margins ought to coincide along an efficient allocation (x_i). Hence

$$\mathcal{U}_1'(x_1) = \mathcal{U}_2'(x_2) = \cdots. \tag{6}$$

Whatever be the specific criteria, (6) begs two immediate questions. First, how is differentiability defined? Second, does $\mathcal{U}_i$ indeed qualify as differentiable at x_i? A most elementary instance already justifies legitimate concerns:

Example 2.1 (*Kinky Objectives*)**.** Suppose producer $i \in I = \{1, 2\}$ gets payoff

$$\mathcal{U}_i(x_i) = m_i \min\{x_i, \bar{x}_i\}$$

from a single production factor, used in the amount $x_i \in [0, +\infty) =: X_i \subset \mathbb{X} := \mathbb{R}$. Given net margins $m_1 > m_2$, production capacities $\bar{x}_1, \bar{x}_2 > 0$, and $\bar{x}_1 < x_I < \bar{x}_1 + \bar{x}_2$, the efficient allocation $(\bar{x}_1, x_I - \bar{x}_1)$ cannot fit (6) because $\mathcal{U}_1$ is not differentiable at $x_1 = \bar{x}_1$.[2] ♢

Besides concerns with differentiability, (6) brings up another worry, no less important, namely: what happens if x_i is not interior to X_i?

Example 2.2 (*Simple But Extreme Allocation*)**.** Let $X_i \subset \mathbb{X} = \mathbb{R}$ be a closed interval. Its upper bound $\bar{x}_i$ is finite but the lower bound $\underline{x}_i$ possibly infinite. For feasibility, presume $\sum_{i \in I} \underline{x}_i \le x_I \le \sum_{i \in I} \bar{x}_i$. Further, let $\mathcal{U}_i$ (or an extension thereof) be differentiable on X_i, not necessarily concave, but

$$\mathcal{U}_1'(x_1) > \mathcal{U}_2'(x_2) > \cdots \quad \text{when } x_1 \in X_1, x_2 \in X_2, \ldots.$$

Since the instance $\sum_{i \in I} \underline{x}_i = x_I$ is trivial, assume $\sum_{i \in I} \underline{x}_i < x_I$. If an efficient allocation has $x_j \in intX_j$ for some (smallest) j, then $x_i = \bar{x}_i$ for all $i < j$, and $x_i = \underline{x}_i > -\infty$ for all $i > j$, so that

$$x_j = x_I - \sum_{i<j} \bar{x}_i - \sum_{i>j} \underline{x}_i.$$

(By convention, an empty sum is nil.) In particular, if $\underline{x}_{|I|} = -\infty$, then $j = |I|$. Otherwise, when no $x_i \in intX_i$, j is the largest index for which

$$x_j = x_I - \sum_{i<j} \bar{x}_i - \sum_{i>j} \underline{x}_i = \bar{x}_j.$$

Plainly, in either case, (6) cannot hold. ♢

Example 2.3 (*Nonstandard Risk Sharing*)**.** Suppose each agent faces future scenario s with probability $\mu_s > 0$, $\sum_{s \in S} \mu_s = 1$, S being finite. Let $\mathbb{X} = \mathbb{R}^S$ encompass all *contingent claims* $s \in S \mapsto x_s \in \mathbb{R}$ to money, and posit Bernoulli objectives

$$\mathcal{U}_i(x_i) := Eu_i(x_i) := \sum_{s \in S} u_i(x_{is})\mu_s$$

when $x_i \in X_i := \mathbb{R}^S_+$. Presuming each u_i concave here, numerous studies have characterized efficient risk sharing (Borch, 1962; Bühlmann, 1984; Cass et al., 1996; Leland, 1978; Malinvaud, 1973). It is commonly recommended that neutral agents, if any, act as insurers. More generally, risk had better be split – at every margin – in proportion to parties' tolerance (Wilson, 1968; Wyler, 1984). Such recommendations hinge, however, on smooth objectives and interior choice. Otherwise they may be misleading. To wit, let agents $i = 1, 2$ have concave, differentiable utility indices

$$r \in \mathbb{R} \mapsto u_1(r) = r \quad \text{and} \quad u_2(r) = -r^2/2 + 3r.$$

If $0 < x_{Is} < 2$ for each s, the efficient allocation has $x_1 = 0$ and $x_2 = x_I$. Thus the neutral (hence infinitely tolerant) agent 1 gets full insurance whereas the averse party 2 carries all risk.[3] ♢

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1 Use of normal cones permits, in a first pass, to postpone various details, many somewhat distracting. Thereby, optimality conditions become geometric; they invoke vectors which point perpendicularly out of the feasible set. For instance, if $\mathbb{X}$ is Euclidean, and $X = \cap_{a \in A} \{x \in \mathbb{X} : a \cdot x \le b_a\}$ for some finite set $A \subset \mathbb{X}$, then $N(X, x) = \left\{\sum_{a \in A} r_a a : 0 \le r_a \perp (b_a - a \cdot x) \ge 0\right\}$ is spanned by the normals to the binding hyperplanes.

2 Little regularity lacks in this example: each agent has concave criterion, closed convex domain, and interior choice. Moreover, along the solution, just one criterion is nonsmooth at just one isolated point. But precisely that point overthrows (6).

3 If one prefers u_2 strictly increasing on $[0, +\infty)$, let $u_2(r) = 2r + 1/2$ when $r > 1$. This modification would not overthrow the above results. If x_I is replaced by another non-negative risk, quite similar conclusions are obtained. Note that *concave preferences are not essential*; the allocation remains the same provided $u_1' < u_2'$ on $(0, +\infty)$.