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Robust optimal risk sharing and risk premia in expanding pools

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ABSTRACT

We consider the problem of optimal risk sharing in a pool of cooperative agents. We analyze the asymptotic behavior of the certainty equivalents and risk premia associated with the Pareto optimal risk sharing contract as the pool expands. We first study this problem under expected utility preferences with an objectively or subjectively given probabilistic model. Next, we develop a robust approach by explicitly taking uncertainty about the probabilistic model (ambiguity) into account. The resulting robust certainty equivalents and risk premia compound risk and ambiguity aversion. We provide explicit results on their limits and rates of convergence, induced by Pareto optimal risk sharing in expanding pools.

1. Introduction

Risk sharing constitutes a main principle in economic and mathematical risk theory. It refers to a subdivision of the aggregate risk in a pool by exchanging and relocating risks among the cooperative individuals that participate in the pool. Risk sharing provides a means of inducing risk reduction for the individuals, in a potentially Pareto optimal sense. Since the seminal work by Borch (1962) it has been studied by numerous authors in a wide variety of settings; see e.g., Arrow (1963), Wilson (1968), DuMouchel (1968), Gerber (1978, 1979), Bühlmann and Jewell (1979), Landsberger and Meilijson (1994), and, more recently, Carlier and Dana (2003), Heath and Ku (2004), Barrieu and El Karoui (2005, 2009), Dana and Scarsini (2007), Jouini et al. (2008), Kiesel and Rüschendorf (2008), Ludkovski and Rüschendorf (2008), Filipović and Svindland (2008), Dana (2011), Ravanelli and Svindland (2014), and the references therein.

This paper explores what happens when Pareto optimal risk sharing is combined with an *expanding* pool of risks. In an expanding pool of independent and identically distributed (i.i.d.) risks, the distribution of the aggregate risk spreads, but the average risk obeys the law of large numbers and converges to its expectation (see e.g., Samuelson (1963), Diamond (1984) and Ross (1999) for a detailed discussion). We analyze when the individuals' risk reduction induced by Pareto optimal risk sharing may be exploited to the full limit: when, upon subdividing and relocating the aggregate risk according to the Pareto optimal risk sharing rule

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in an expanding pool of i.i.d. risks with cooperating individuals that have identical preferences, will risk sharing eventually lead to annihilating risk beyond its expectation?

We answer this question by analyzing, in a general setting, the asymptotic behavior of the certainty equivalents and risk premia in an expanding pool of risks under Pareto optimal risk sharing. Adopting the classical expected utility model of Von Neumann and Morgenstern (1944), Pratt (1964) studies the connection between the risk premium, defined as the expected value of a given risk minus its certainty equivalent (the monetary amount that makes an agent indifferent to the risk), and the utility function. He shows that greater local risk aversion (risk aversion in the small) at all wealth levels implies greater global risk aversion (risk aversion in the large) and vice versa, in the sense that the risk premia vary with the local risk aversion intensity. Furthermore, Pratt (1964) provides an expansion of the risk premium for a small and actuarially fair risk, given by the local risk aversion times half the variance of the risk. Hence, with vanishing variance of the average risk in an expanding pool of i.i.d. risks – as implied by the law of large numbers –, the risk premium associated with the Pareto optimal risk sharing rule can be seen to vanish, too. We analyze this convergence rigorously and derive results on the risk premium's rate of convergence. We first consider the relatively simple case of the expected utility model, as in Pratt (1964) but with refined results, and next turn to more advanced decision models, for which the problem proves to be much more delicate.

In recent years, the distinction between risk (probabilities given) and ambiguity (probabilities unknown) has received much attention. Under Savage (1954) subjective expected utility model this distinction is absent due to the assignment of subjective probabilities. Modeling approaches that explicitly recognize the fact that a specific probabilistic model may be misspecified are referred to as robust (see e.g., Hansen and Sargent (2001, 2007)). A popular class of models for decision under risk and ambiguity is provided by the multiple priors models (Gilboa and Schmeidler, 1989; see also Schmeidler, 1986, 1989). It occurs as a special case of the rich variational and homothetic preference models (Maccheroni et al., 2006; Cerreia-Vioglio et al., 2011; Chateauneuf and Faro, 2010). These models all reduce to the expected utility model of Von Neumann and Morgenstern (1944) when ambiguity has resolved in the classical Anscombe and Aumann (1963) setup. A related strand of literature in financial mathematics is that of convex measures of risk introduced by Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), and Heath and Ku (2004); see also the early Wald (1950), Huber (1981), Deprez and Gerber (1985), Ben-Tal and Teboulle (1986, 1987), and the more recent Carr et al. (2001), Ruszczyński and Shapiro (2006), Ben-Tal and Teboulle (2007) and Goovaerts et al. (2011). Föllmer and Schied (2011, 2013) and Laeven and Stadje (2013, 2014) provide precise connections between the two strands of the literature. We explore the combination of optimal risk sharing and an expanding pool of risks in the presence of uncertainty about the true probabilistic model.

More specifically, we start in this paper by considering classical expected utility, so that the certainty equivalent U of a risk X is given by

$$U(X) = u^{-1}\left(\mathrm{E}\left[u(X)\right]\right), \tag{1.1}$$

with u a utility function and $\mathrm{E}[\cdot]$ the expectation under an objectively or subjectively given probabilistic model. We analyze in this setting the precise limiting behavior and convergence rates of the risk premium associated with the average risk S_n/n, where $S_n = \sum_{i=1}^{n} X_i$ for i.i.d. risks X_i, $n \in \mathbb{N}$, given by

$$\pi(v, S_n/n) = \mathrm{E}\left[v + S_n/n\right] - U(v + S_n/n), \tag{1.2}$$

corresponding to proportional (equal, $1/n$) risk sharing of the aggregate risk among n cooperative individuals with identical utility function u and initial wealth v, which we prove to be Pareto optimal in this setting under mild conditions.

Next, we explicitly take uncertainty about the probabilistic model into account and adopt a robust approach. This setting turns out to be intriguingly more delicate. It is best thought of as featuring probabilistic models that are the Kolmogorov extensions of a family of product probability measures. We first consider certainty equivalents that are "robustified" over a class of such probabilistic models $\mathcal{P}$:

$$\mathcal{U}_{\mathcal{P}}(X) = \inf_{Q\in\mathcal{P}} U_Q(X) + \alpha(Q), \tag{1.3}$$

with $U_Q(X) = u^{-1}\left(\mathrm{E}_Q\left[u(X)\right]\right)$ and where $\alpha : \mathcal{P} \to \mathbb{R} \cup \{\infty\}$ is a penalty function that measures the plausibility of the probabilistic model $Q \in \mathcal{P}$. We prove that the proportional risk sharing rule remains Pareto optimal in this setting. Furthermore, we prove that in an expanding pool of risks the robustified certainty equivalent of the average risk converges to the robustified expectation, and we provide explicit bounds on the corresponding convergence rates. We find in particular that the convergence rates are dictated by the individuals' coefficient of absolute risk aversion and the robustified first two moments, expectation and variance.

Finally, we naturally extend the risk premium of Pratt (1964) to our setting with risk *and* ambiguity, by considering

$$\pi(v, X) = \mathcal{W}(v + X) - \mathcal{W}_{\mathcal{P}}(v + X) \quad \text{and}$$
$$\pi(v, X) = \mathcal{U}(v + X) - \mathcal{V}_{\mathcal{P}}(v + X),$$

in the case of homothetic and variational preferences, respectively, with

$$\mathcal{W}(X) = \inf_{Q\in\mathcal{P}} \mathrm{E}_Q\left[X\right]\beta(Q) \quad \text{and}$$

$$\mathcal{W}_{\mathcal{P}}(X) = u^{-1}\left(\inf_{Q\in\mathcal{P}} \mathrm{E}_Q\left[u(X)\right]\beta(Q)\right),$$

where $\beta : \mathcal{P} \to [1, \infty]$ is a penalty function, and

$$\mathcal{U}(X) = \inf_{Q\in\mathcal{P}} \mathrm{E}_Q\left[X\right] + \alpha(Q) \quad \text{and}$$

$$\mathcal{V}_{\mathcal{P}}(X) = u^{-1}\left(\inf_{Q\in\mathcal{P}} \mathrm{E}_Q\left[u(X)\right] + \alpha(Q)\right).$$

The robustified certainty equivalents and risk premia compound risk and ambiguity aversion (Ghirardato and Marinacci, 2002). We prove that under Pareto optimal risk sharing in an expanding pool of risks the robust risk premium converges to zero in the homothetic case, but, for non-trivial α, will not vanish in the limit in the variational case, in which case it converges to $\mathcal{U}(v + X_1) - \mathcal{V}(v + X_1)$, with

$$\mathcal{V}(X) = u^{-1}\left(\inf_{Q\in\mathcal{P}} u\left(\mathrm{E}_Q\left[X\right]\right) + \alpha(Q)\right),$$

and we analyze the corresponding convergence rates.

Our convergence results may be compared to the convergence results obtained by Föllmer and Knispel (2011a,b). These authors analyze the limiting behavior of the risk capital per financial position, when computing capital requirements for large and expanding portfolios of i.i.d. financial positions, in the *absence* of optimal risk sharing (and in the more restrictive setting of convex measures of risk rather than the general setting provided by homothetic and variational preferences, as is considered here). This seemingly related problem requires much different techniques and leads to completely different results. For example, without optimal risk sharing, the certainty equivalent per position in an expanding portfolio of n i.i.d. risks under expected utility with exponential utility (which yields a prototypical example of