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The role of a representative reinsurer in optimal reinsurance

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ABSTRACT

In this paper, we consider a one-period optimal reinsurance design model with n reinsurers and an insurer. For very general preferences of the insurer and that all reinsurers use a distortion premium principle, we establish the existence of a representative reinsurer and this in turn facilitates solving the optimal reinsurance problem with multiple reinsurers. The insurer determines its optimal risk that it wants to reinsure via this pricing formula. The risk to be reinsured is then shared by the reinsurers via tranching. The optimal ceded loss functions among multiple reinsurers are derived explicitly under the additional assumptions that the insurer's preferences are given by an inverse-S shaped distortion risk measure and that the reinsurers' premium principles are some functions of the Conditional Value-at-Risk. We also demonstrate that under some prescribed conditions, it is never optimal for the insurer to cede its risk to more than two reinsurers.

1. Introduction

The optimal risk sharing between an insurer and a reinsurer is one of the most challenging problems that has been heavily studied in the academic literature and actuarial practice. This problem is first formally analyzed by Borch (1960) who demonstrates that, under the assumption the reinsurance premium is calculated by the expected value principle, the stop-loss reinsurance treaty is the optimal strategy that minimizes the variance of the retained loss of the insurer. By maximizing the expected utility of the terminal wealth of a risk-averse insurer, Arrow (1963) similarly shows that the stop-loss reinsurance treaty is optimal. These pioneering results have subsequently been refined to incorporating more sophisticated optimality criteria and/or more realistic premium principles. See, for example, Kaluszka (2005) and Chi and Tan (2011) for a small sample of these generalizations. These results indicate that more exotic strategies such as that based on the limited stop-loss or truncated stop-loss could be optimal, as opposed to the stop-loss reinsurance.

While most of the existing literature on optimal reinsurance have predominantly confined to analyzing the optimal risk sharing between two parties, i.e., an insurer and a reinsurer, recently some progress has been made on addressing the optimal reinsurance in the presence of multiple reinsurers. See, for example, Asimit et al. (2013), Chi and Meng (2014), and Cong and Tan (2016). Such formulation is more reasonable since in a well established reinsurance market, an insurer could always use more than one reinsurer to reinsure its risk. In fact it may be desirable for the insurer to do so in view of the differences in reinsurers' risk attitude and the competitiveness of the reinsurance market. Some reinsurers may have higher risk tolerance and maybe more aggressive in pricing certain layers of risk. As a result, insurer that exploits such discrepancy among reinsurers might be able to achieve better risk sharing profile.

Motivated by these results, this paper studies the problem of optimal reinsurance in the presence of multiple reinsurers. The significant contributions of our proposed study can be described as follows. First, we allow for very general preferences of the insurer. In contrast, Cong and Tan (2016) assume that the insurer's objective is to minimize its value at risk (VaR), while both Asimit et al. (2013) and Chi and Meng (2014) assume conditional value at risk (CVaR), in addition to VaR. Second, we allow for more than two reinsurers while the optimal reinsurance models of Asimit et al. (2013) and Chi and Meng (2014) explicitly assume two reinsurers. Third, both Asimit et al. (2013) and Chi and Meng (2014) impose the condition that one of the reinsurers adopts the expected value premium principle. Our proposed model, on the other hand, does not have such constraint. In fact, our premium principle is quite general in that we assume the reinsurers use distortion premium principles. Many authors, including Wang (1995, 2000), Wang et al. (1997), De Waegenaere et al. (2003), Chen and Kulperger (2006),

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Cheung (2010), Cui et al. (2013), and Assa (2015), use distortion functions to price risk. Special cases include the pricing principles induced by Wang transform, VaR, and expected value principle. Fourth, we also analyze the uniqueness of proposed solution. Finally, we demonstrate that under some additional assumptions, it is never optimal for an insurer to cede its loss to more than two reinsurers.

If there is only one reinsurer and the insurer maximizes dual utility (Yaari, 1987), the optimal reinsurance contract is given by tranching of the total insurance risk as shown by Cui et al. (2013), and Assa (2015). We extend this result to the case of multiple reinsurers and show that the optimal ceded loss functions could be in the form of multiple tranches, with each tranche being allocated to each reinsurer.

The layout of the remaining paper is as follows. The model setup is described in Section 2. In Section 3, we show our main results that characterize the representative reinsurer. Section 4 describes the optimal reinsurance contracts if the insurer uses dual utility. Section 5 provides an example where reinsurance prices are determined via the well-known CVaR. Finally, Section 6 concludes the paper.

2. Model setup

The purpose of this section is to describe various important concepts including the distortion premium principles and our proposed formulation of the optimal reinsurance model in the presence of multiple reinsurers. These are described in Sections 2.1 and 2.2, respectively.

2.1. Distortion premium principles

In this subsection, we introduce the pricing formula of the reinsurers. Let $(\Omega, F, \mathbb{P})$ be a probability space and $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the class of bounded random variables on it. For brevity, we use the notation L^∞ to denote $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ when there is no confusion. We interpret random variables as a loss. We now define the distortion risk measure, which is due to Wang et al. (1997):

Definition 2.1. The distortion risk measure is given by

$$\rho^g(Z) = \int_0^\infty g(S_Z(z))dz, \quad \text{for all } Z \in L^\infty, \tag{1}$$

where S_Z is the survival function of the loss Z and the probability distortion function $g : [0, 1] \to [0, 1]$ is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$.

Corresponding to the distortion risk measure, we have the distortion premium principle, which is defined as follows:

Definition 2.2. The distortion premium principle is given by

$$\pi^{\theta,g}(Z) = (1 + \theta) \cdot \rho^g(Z), \quad \text{for all } Z \in L^\infty, \tag{2}$$

where $\theta \geq 0$ can be interpreted as the loading factor and g is a probability distortion function.

Note that when $g(x) = x$, the above distortion premium principle reduces to the (loaded) expected value premium principle. When the distortion function is concave, the distortion premium principle recovers Wang's premium principle. Wu and Wang (2003) and Wu and Zhou (2006) provide a characterization of the distortion premium principle based on additivity of comonotonic risks. The premium principle formulation (2) allows also for pricing formulas that include a risk component such as $\pi(Z) = E[Z] + \alpha \cdot \rho^{\hat{g}}(Z)$ for $\alpha \geq 0$, where $\rho^{\hat{g}}$ can be a VaR or a CVaR. See, for example, Acerbi and Tasche (2002) and (24) or (26) in Section 5.

2.2. Reinsurance model set-up

We now turn to our proposed optimal reinsurance model. We assume that an insurer faces a non-negative and bounded random loss $X \in L^\infty$ and that $M = \operatorname{esssup} X = \inf\{a \in R : \mathbb{P}(X > a) = 0\}$. We further assume that there are n reinsurers in this market and that each of these reinsurers is indexed by $i \in \{1, \ldots, n\}$. The probability distribution of the loss exposure X of the insurer is assumed to be a common knowledge to all the participating reinsures and let $f_i(X)$ represent the portion of the loss X that is ceded to reinsurer i. The problem of optimal reinsurance is therefore concerned with the optimal partitioning of X into $f_i(X), i = 1, \ldots, n$, and $X - \sum_{i=1}^n f_i(X)$. Note that $\sum_{i=1}^n f_i(X)$ represents the aggregate loss that is ceded to all n participating reinsurers so that $X - \sum_{i=1}^n f_i(X)$ captures the loss that is retained by the insurer.

Let us now provide some additional discussion on the shape of ceded loss functions. In particular, the set of ceded loss functions that is of interest to us is of the following:

$$\mathcal{F} = \Big\{ f : [0, M] \to [0, M] \Big| f(0) = 0, \\ 0 \leq f(x) - f(y) \leq x - y, \ \forall\, 0 \leq y < x \leq M \Big\}. \tag{3}$$

While in the reinsurance market the ceded loss function admits a variety of forms, virtually all of these contracts satisfy the property as stipulated in $\mathcal{F}$. The importance for a ceded loss function to satisfy $\mathcal{F}$ is driven by the concern with moral hazard, which otherwise will exist for the insurer (see, e.g., Denuit and Vermandele, 1998; Young, 1999; Bernard and Tian, 2009). For this reason, we similarly require that $f_i(X), i = 1, \ldots, n$, and $X - \sum_{i=1}^n f_i(X)$ belong to $\mathcal{F}$ (see Asimit et al., 2013; Chi and Meng, 2014). See also Remark 4 at the end of the next section.

For any $f \in \mathcal{F}$, we have

$$\rho^g(f(X)) = \int_0^M g(S_X(z))df(z). \tag{4}$$

The above result follows from the definition of the distortion risk measure (1), $f(0) = 0$ and the fact that f is 1-Lipschitz. See Zhuang et al. (2016, Lemma 2.1 therein), for a detailed proof.

By partially transferring part of the loss to reinsurers, the insurer incurs an additional cost in the form of reinsurance premium that is payable to the reinsurers. The reinsurance premium depends on the ceded loss function $f_i(X)$, the loading factor θ_i, and the probability distortion function g_i of the reinsurer. We use $\pi^{\theta_i,g_i}(f_i(X))$ to denote the resulting reinsurance premium that is charged by the reinsurer i for assuming loss $f_i(X)$. Let W denote the future wealth of the insurer in the absence of insuring risk X. The future wealth W can be random or deterministic. By insuring and reinsuring X, the net worth of the insurer becomes $W - X + \sum_{i=1}^n f_i(X) - \sum_{i=1}^n \pi^{\theta_i,g_i}(f_i(X))$. Given that the reinsurance premium increases with the ceded losses, this suggests that a conservative insurer could eliminate most of its risk at the expense of higher reinsurance premium. On the other hand, a more aggressive insurer could reduce its reinsurance premium but exposes itself to a higher potential loss. This demonstrates the trade-off between risk retention and risk transfer.

From a risk management point of view, the existence of such a trade-off also implies that it is important for the insurer to seek the best reinsurance strategy that optimally balances between risk retention and risk transfer. This can be accomplished by formulating the problem as an optimization problem. More specifically, let V capture the utility of an insurer's net worth. Here, V is a function that maps random variables to real numbers. We assume that $V(W) < \infty$, and V is strictly monotonic; i.e., for all