Insurance: Mathematics and Economics 70 (2016) 320–326

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Long-term behavior of stochastic interest rate models with Markov switching

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ARTICLE INFO

Article history:
Received December 2015
Received in revised form
June 2016
Accepted 28 June 2016
Available online 11 July 2016

MSC:
60H10
37A25

Keywords:
Cox–Ingersoll–Ross (CIR) model
Markov chain
Stationary distribution
Feller property
Hölder continuous

ABSTRACT

In this paper, we consider the long time behavior of Cox–Ingersoll–Ross (CIR) interest rate model with Markov switching. Using the ergodic theory of switching diffusions, we show that CIR model with Markov switching has a unique stationary distribution. Furthermore, we prove that the sequence $\overline{X}_t := \frac{1}{t}\int_0^t X_s ds$ converges almost surely. As a by-product, we find that the marginal stationary distribution for CIR model with Markov switching can be determined uniquely by its moments.

1. Introduction

An interest rate is the rate at which interest is paid by borrowers for the use of money that they borrow from lenders. To characterize the interest rate, many mathematical models are proposed to characterize the short-term borrowing. If the interest rate is determined by only one stochastic differential equation, the model is called one-factor model. Cox et al. (1985) assume that the evolution of the interest rate is given by

$$dX_t = (\delta - \beta X_t)dt + \sigma\sqrt{X_t}dW_t, \tag{1.1}$$

where δ, β, σ are positive constants, W_t is a standard (one-dimensional) Brownian motion. The SDE defined by (1.1) is said to be CIR model. It is well known that the CIR model (1.1) is nonnegative and has some empirically relevant properties. In this model, the interest rate X_t has a unique stationary (steady state) distribution, which follows a gamma distribution, denoted by $\Gamma(\frac{2\delta}{\sigma^2}, \frac{2\beta}{\sigma^2})$. The rate displays the mean reversion towards the long-term constant $\frac{\delta}{\beta}$ (the mean of stationary distribution, denoted by EX_∞). The CIR model has been widely used in finance and insurance.

However, as a one factor model, the primary shortcoming is that it cannot capture complicated the yield curve behavior; It tends to produce parallel shifts in the yield curve, but not changes in its slope or curvature. To better overcome the shortcoming and capture the empirical data, Longstaff and Schwartz (1992) proposes a two-factor interest rate model, which is assumed that drift coefficients and diffusion coefficients are determined by two CIR-type stochastic differential equations. Chen proposes a three-factor interest rate model. Deelstra and Delbaen (1995) extend CIR model (1.1) to the following stochastic differential equation (SDE)

$$dX_t = (2\beta X_t + \delta_t)dt + g(X_t)dW_t, \tag{1.2}$$

where $\delta_t(\omega)$ is a positive continuous adapted process, and $g : \mathbb{R} \to \mathbb{R}^+$ is a function, vanishing at zero and such that there is a constant b with $|g(x) - g(y)| \le b\sqrt{|x-y|}$. Under some assumptions, they show that $\frac{1}{t}\int_0^t X_s ds$ tends to a constant. Zhao (2009) considers the long-term behavior of the model (1.2) with compound Poisson jumps. Bao and Yuan (2013) study the long time behavior of the model (1.2) with delay and compound Poisson jumps.

However, empirical research has indicated that diffusion processes with Markov switching can better capture the reality data. For example, Ball and Torous (1999) argue that short term interest rates should be characterized by a nonlinear regime-shifting model to account for a change in economic regime brought

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http://dx.doi.org/10.1016/j.insmatheco.2016.06.017
0167-6687/

about by factors such as Federal Reserve experiment in the early 1980s and OPEC oil crises in the late 1970s. Smith (2002) finds that diffusion with Markov switching model is more reasonable to model the monthly rate treasury bill of USA from June 1964 to December 1996.

In this paper, we consider an interest rate model satisfying the following stochastic differential equation (SDE)

$$dX_t = (2\beta(r_t)X_t + \delta(r_t))dt + g(X_t, r_t)dW_t \tag{1.3}$$

with $\beta(i) < 0, i \in \mathbb{S}$, and r_t is a Markov chain. We call model (1.3) the CIR model with Markov switching. Here $g : \mathbb{R} \times \mathbb{S} \to \mathbb{R}^+$ such that there is a constant b with $|g(x, i) - g(y, j)| \le b\sqrt{|x - y|}$. Besides, $g(0, i) \equiv 0$, for any $i \in \mathbb{S}$.

Although there is an extensive literature on quantitative and qualitative properties of the generalized CIR models, to the best of the authors knowledge, there are few papers discussing the theoretical properties of CIR model (1.3). The main purpose of this paper is to investigate the long time behavior of the CIR model (1.3). We show that the model has a unique stationary distribution. Moreover, we prove that the sequence $\overline{X}_t := \frac{1}{t}\int_0^t X_s ds$ converges almost surely. Besides, we show that the marginal stationary distribution of CIR model (1.3) can be determined uniquely by its moments.

2. Notation

Throughout this paper, we let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathbb{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathbb{F}_0$ contains all $\mathbb{P}$-null sets). Let W_t be a Brownian motion and r_t be a Markov chain taking the state space $\mathbb{S} = \{1, \dots, m_0\}$ with generator $\Gamma = (\gamma_{ij})_{m_0\times m_0}$ given by

$$P\{r_t = j | r_0 = i\} = \begin{cases} \gamma_{ij} t + o(t) & \text{if } i \ne j \\ 1 + \gamma_{ii} t + o(t) & \text{if } i = j. \end{cases} \tag{2.1}$$

Here the γ_{ij} represents the transition rate from i to j, and $\gamma_{ij} > 0$ if $i \ne j$, while

$$\gamma_{ii} = -\sum_{i\ne j} \gamma_{ij}. \tag{2.2}$$

In addition, we assume that m_0 is a finite natural number. Note that the assumptions (2.1), (2.2) and $\gamma_{ij} > 0$ for $i \ne j, i, j \in \mathbb{S}$ imply that the Markov chain r_t is irreducible. Besides, it is not hard to see that r_t with finite state $\mathbb{S}$ is ergodic. Let $\pi_i := \lim_{t\to+\infty} P(r_t = i | r_0 = j),\ i \in \mathbb{S}$. If $m_0 = 2$, then

$$\pi_i = \frac{\frac{1}{\gamma_{ii}}}{\sum_{j=1}^{m_0} \frac{1}{\gamma_{jj}}}.$$

Throughout this paper, the following notation is frequently used.

K: denoting a generic positive constant whose values may vary at its different appearances.
$\mathbb{R}_{\ge 0}$: $[0, +\infty)$, i.e. all nonnegative real numbers.
$\mathbb{N}^+$: standing for all positive integers.
$X_t^{x,i}$: standing for the interest rate process X_t with initial value $X_0 = x, r_0 = i$.

For any bounded open interval (or bounded left closed right open interval) $D \subset \mathbb{R}_{\ge 0}, i \in \mathbb{S}$, let

$$\tau_D = \inf\{t : X_t^{x,i} \in D\}.$$

If for any $x \in D^c := \mathbb{R}_{\ge 0} - D, E\tau_D < +\infty$, then the process $X_t^{x,i}$ is said to be positive recurrent with respect to D. If the process $X_t^{x,i}$ is positive recurrent relative to any bounded open set $D \subset \mathbb{R}_{\ge 0}$, then the process $X_t^{x,i}$ is positive recurrent.

To guarantee that there exists a unique nonnegative solution, we impose the following assumptions (A):

(A1) $\delta(i) > 0, \beta(i) < 0, i \in \mathbb{S}$;
(A2) $g(x, i)$ is a nonnegative function with the following properties:
  (1) $g(0, i) = 0, i \in \mathbb{S}$;
  (2) There is a constant K such that for all $x, y \in \mathbb{R}^+, i \in \mathbb{S}$, $|g(x, i) - g(y, i)| \le K\sqrt{|x - y|}$.

Let Δ_{ij} be consecutive (with respect to the lexicographic ordering on $\mathbb{S} \times \mathbb{S}$), left closed, right open intervals of the real line, each having length γ_{ij} (see, e.g., pp. 46–48, Mao and Yuan (2006)).

Define a function

$$h(i, y) = \begin{cases} j - i & y \in \Delta_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

The Markov chain $r(t)$ can be rewritten as

$$dr(t) = \int_{\mathbb{R}} h(r(t-), y) N(dt, dy),$$

where $N(dt, dy)$ is Poisson random measure, with random measure $dt \times m(dy)$, in which $m(dy)$ is the Lebesgue measure on $\mathbb{R}$. And for simplicity, we let

$$\widetilde{N}(dt, dy) = N(dt, dy) - dt \times m(dy),$$

which is the martingale measure.

3. Preliminaries

In this section, we shall consider existence and uniqueness and several properties for the interest model (1.3).

Lemma 3.1. *Let assumption* (A) *hold. Then the SDE defined by* (1.3) *has a unique nonnegative solution.*

Proof. Recall that the Markov chain can be rewritten as

$$r_t = r_0 + \sum_{n=1}^{\infty} Z_n I(\tau_n \le t), \tag{3.1}$$

where τ_n, Z_n have the following conditional distributions. Given that $r(\tau_k) = i$, $\tau_{k+1} - \tau_k$ follows exponential distributed with mean γ_{ii}^{-1}, and the jump $Z_{k+1} = r(\tau_{k+1}) - r(\tau_k)$ is independent of the past and has a probability of $-\frac{\gamma_{ij}}{\gamma_{ii}}$. According to Deelstra and Delbaen (1995), there exists a (pathwise) unique nonnegative solution to the equation, for each i,

$$Y_t = y + \int_0^t (2\beta(i)Y_s + \delta(i))ds + \int_0^t g(Y_s, i)dW_s, \quad Y_0 = y.$$

Note the unique solution of Y_t is a diffusion process without Markov switching. To emphasize the diffusion Y_t with different diffusion and drift coefficients, for given i, we denote the unique solution Y_t by $Y_t^{y,i}$. For each $k \in \mathbb{N}$, $t \in [\tau_k, \tau_{k+1})$, we have $r_t = j \in \mathbb{S}$. Thus, we obtain a sequence of nonnegative solution $\{Y_t^{Y_{\tau_k}, r_{\tau_k}}\}_{k\in\mathbb{N}}$. Based on the sequence of nonnegative solutions, we construct the solution to the model (1.3) as follows. With a bit of abuse of notation, define, on $t \in [0, \tau_1)$

$$X_t = Y_t^{X_0, i}.$$

Then, for $\tau_1 \le t \le \tau_2$, we define

$$X_t = Y_t^{Y_{\tau_1}, r_{\tau_1}}.$$