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Sufficient conditions for ordering aggregate heterogeneous random claim amounts

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ABSTRACT

This note has a revisit to stochastic comparison on aggregate claim amounts. We develop sufficient conditions for the usual stochastic order on aggregate claim amounts of independent claim sizes and with a common occurrence frequency vector. Besides, we obtain the usual stochastic order on aggregate claim amounts with a common WSAI claim size vector, and this also improves Theorem 4.6 of Zhang and Zhao (2015).

1. Introduction

For two n dimensional real vectors $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$, denote $\mathcal{I}_n = \{1, \ldots, n\}$, $\mathcal{D}_n = \{\boldsymbol{\lambda} : \lambda_1 \geq \cdots \geq \lambda_n\}$ and $\mathcal{D}_n^+ = \{\boldsymbol{\lambda} : \boldsymbol{\lambda} \in \mathcal{D}_n,\ \lambda_i > 0,\ i \in \mathcal{I}_n\}$, and let

$$\mathcal{M}_n = \left\{ \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} : \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{D}_n^+ \right\}$$

be the set of $2 \times n$ real matrices with each row having decreasing components and

$$\mathcal{U}_n = \left\{ \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} : \lambda_i, \mu_j > 0 \text{ for all } i, j \in \mathcal{I}_n \text{ and } (\lambda_i - \lambda_j)(\mu_i - \mu_j) \geq 0 \text{ for any } \{i, j\} \subset \mathcal{I}_n \right\}$$

the one with components in both rows being arrayed in the same pattern. It is plain that $\mathcal{M}_n \subseteq \mathcal{U}_n$. For $\boldsymbol{p} = (p_1, \ldots, p_n)$, $\boldsymbol{q} = (q_1, \ldots, q_n)$ and a real function g, denote

$$\boldsymbol{p}_g = \big(g(p_1), \ldots, g(p_n)\big), \qquad \boldsymbol{q}_g = \big(g(q_1), \ldots, g(q_n)\big).$$

Suppose $(I_{p_1}, \ldots, I_{p_n})$ and $(I_{q_1}, \ldots, I_{q_n})$ are two groups of mutually independent Bernoulli random variables (denoted as $(I_{p_1}, \ldots, I_{p_n}) \sim \mathrm{Ber}(\boldsymbol{p})$ and $(I_{q_1}, \ldots, I_{q_n}) \sim \mathrm{Ber}(\boldsymbol{q})$) with $\mathsf{E}[I_{p_i}] = p_i$, $\mathsf{E}[I_{q_i}] = q_i$ for $p_i, q_i \in (0, 1)$ and $i \in \mathcal{I}_n$, and nonnegative random variables $(X_1, \ldots, X_n)$ are independent of $(I_{p_1}, \ldots, I_{p_n})$ and $(I_{q_1}, \ldots, I_{q_n})$. In insurance, X_i's represent claim sizes of risks covered by one policy, I_{p_i}'s are indicators of occurrences of these claims, thus $\sum_{i=1}^n I_{p_i}$ gives the total number of claims and $\sum_{i=1}^n I_{p_i} X_i$ defines the aggregate claim amounts in the portfolio.

In the past decades, lots of researchers paid their attention to ordering the total number of claims. Karlin and Novikoff (1963) took the first to show that

$$\boldsymbol{q} \overset{\mathrm{m}}{\preceq} \boldsymbol{p} \Longrightarrow \sum_{i=1}^n I_{p_i} \leq_{\mathrm{cx}} \sum_{i=1}^n I_{q_i},$$

where '$\overset{\mathrm{m}}{\preceq}$' and '$\leq_{\mathrm{cx}}$' denote the majorization and the convex order, respectively. Afterward, Pledger and Proschan (1971) and Proschan and Sethuraman (1976) further proved that, for $g(x) = -\ln x$ or $g(x) = (1 - x)/x$,

$$\boldsymbol{q}_g \overset{\mathrm{m}}{\preceq} \boldsymbol{p}_g \Longrightarrow \sum_{i=1}^n I_{q_i} \leq_{\mathrm{st}} \sum_{i=1}^n I_{p_i}, \tag{1.1}$$

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where '$\le_{st}$' denotes the usual stochastic order. Recently, Xu and Balakrishnan (2011) got the following generalization of (1.1),

$$\boldsymbol{q}_g \preceq^{w} \boldsymbol{p}_g \Longrightarrow \sum_{i=1}^{n} I_{q_i} \le_{rh} (\le_{lr}) \sum_{i=1}^{n} I_{p_i},$$
$$\text{for } g(x) = -\ln x \left(g(x) = (1-x)/x\right),$$

where '$\preceq^{w}$', '$\le_{rh}$' and '$\le_{lr}$' denote the weak supermajorization, the reversed hazard rate order and the likelihood ratio order, respectively.

Meanwhile, a lot of work have been done in the literature in pursuit of ordering the aggregate claim amounts from independent claim sizes. According to Ma (2000, Theorem 7), for mutually independent $(X_1, \ldots, X_n)$ with $X_1 \le_{st} \cdots \le_{st} X_n$,

$$\boldsymbol{q}_g \overset{m}{\preceq} \boldsymbol{p}_g \Longrightarrow \sum_{i=1}^{n} I_{q_i} X_i \le_{st} \sum_{i=1}^{n} I_{p_i} X_i, \quad \boldsymbol{q}_g, \boldsymbol{p}_g \in \mathscr{D}_n^{+},$$

where $g(x) = -\ln x$ or $g(x) = (1-x)/x$. For mutually independent nonnegative random variables $(X_{\lambda_1}, \ldots, X_{\lambda_n})$ with $X_{\lambda_i} \sim F(x, \lambda_i)$, a distribution function with parameter λ_i, $i \in \mathcal{I}_n$ and $x \ge 0$, we denote $(X_{\lambda_1}, \ldots, X_{\lambda_n}) \sim \mathcal{L}(F, \boldsymbol{\lambda})$ for short. Let $(I_{p_1}, \ldots, I_{p_n}) \sim \text{Ber}(\boldsymbol{p})$ be independent of $(X_{\lambda_1}, \ldots, X_{\lambda_n}) \sim \mathcal{L}(F, \boldsymbol{\lambda})$ and $(I_{q_1}, \ldots, I_{q_n}) \sim \text{Ber}(\boldsymbol{q})$ be independent of $(X_{\mu_1}, \ldots, X_{\mu_n}) \sim \mathcal{L}(F, \boldsymbol{\mu})$. In the following framework,

- $\bar{F}(x, \lambda)$ is decreasing and convex with respect to λ,
- the survival function of $\sum_{i=1}^{n} X_{\lambda_i}$ is Schur-convex in $\boldsymbol{\lambda}$, and
- $g(x) = -\ln x$ or $g(x) = (1-x)/x$,

Khaledi and Ahmadi (2008) proved that

$$\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{q}_g \end{pmatrix} \ll \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{p}_g \end{pmatrix} \Longrightarrow \sum_{i=1}^{n} I_{q_i} X_{\mu_i} \le_{st} \sum_{i=1}^{n} I_{p_i} X_{\lambda_i},$$
$$\text{for } \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{q}_g \end{pmatrix}, \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{p}_g \end{pmatrix} \in \mathscr{M}_n, \tag{1.2}$$

where '$\ll$' denotes the chain majorization. Recently, Barmalzan et al. (2015) successfully generalized (1.2) to

$$\begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{q}_g \end{pmatrix} \ll \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{p}_g \end{pmatrix} \Longrightarrow \sum_{i=1}^{n} I_{q_i} X_{\mu_i} \le_{st} \sum_{i=1}^{n} I_{p_i} X_{\lambda_i},$$
$$\text{for } \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{q}_g \end{pmatrix}, \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{p}_g \end{pmatrix} \in \mathscr{U}_n. \tag{1.3}$$

Shortly afterwards, Zhang and Zhao (2015, Theorem 3.5) further verified that

$$\boldsymbol{q}_g \preceq^{w} \boldsymbol{p}_g \quad \text{and} \quad \boldsymbol{\mu} \preceq^{w} \boldsymbol{\lambda} \Longrightarrow \sum_{i=1}^{n} I_{q_i} X_{\mu_i} \le_{st} \sum_{i=1}^{n} I_{p_i} X_{\lambda_i},$$
$$\text{for } \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{q}_g \end{pmatrix}, \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{p}_g \end{pmatrix} \in \mathscr{U}_n. \tag{1.4}$$

On the other hand, some other authors take the interdependence among claim sizes into account in ordering the aggregate claim amounts. For exchangeable claim sizes, Ma (2000) derived the convex order and the usual stochastic order of the aggregate claim amounts, and Hu and Ruan (2004) compared the aggregate claim amounts by means of multivariate usual and symmetric stochastic order. For $(X_1, \ldots, X_n)$ with an AI joint density, Zhang and Zhao (2015, Theorem 4.6) obtained that, for $g(x) = -\ln x$ or $g(x) = (1-x)/x$,

$$\boldsymbol{q}_g \overset{m}{\preceq} \boldsymbol{p}_g \Longrightarrow \sum_{i=1}^{n} I_{q_i} X_i \le_{st} \sum_{i=1}^{n} I_{p_i} X_i, \quad \boldsymbol{q}_g, \boldsymbol{p}_g \in \mathscr{D}_n^{+}. \tag{1.5}$$

Along this line of research, the present paper further exploits sufficient conditions for some stochastic orders on aggregate claim amounts. In the context of claim sizes having increasing and concave survival functions with respect to parameter and sum of any two of them having Schur-concave survival function, we study the usual stochastic order on aggregate claim amounts with a random occurrence frequency vector, which serves as a duality of Theorem 3.5 of Zhang and Zhao (2015). Also, we successfully improve the sufficient condition on the usual stochastic order of aggregate claim amounts due to Theorem 4.6 of Zhang and Zhao (2015) by relaxing the AI joint density of the claim sizes to the WSAI claim sizes.

The rest of this paper is laid out as follows: Section 2 comprises of some preliminaries including concerned stochastic orders, majorization order of real vectors, chain majorization, three multivariate dependence notions and a useful lemma. Section 3 investigates sufficient condition for the usual stochastic order between aggregate claim amounts with a common occurrence frequency vector. In Section 4, we present a sufficient condition for the usual stochastic order between aggregate claim amounts of WSAI claim sizes with different occurrence frequency vectors.

From now on, we denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = (0, +\infty)$ and $\mathbb{R}^n = (-\infty, +\infty)^n$. For a matrix A, let A^T be its transpose. For a random variable X, the support, the left and right endpoints of the support are respectively denoted as $\text{supp}(X)$, $l_X = \inf\{x : x \in \text{supp}(X)\}$ and $u_X = \sup\{x : x \in \text{supp}(X)\}$. Throughout this note, all random variables are assumed to be nonnegative, and all expectations are finite whenever they appear. Also the terms *increasing* and *decreasing* mean *nondecreasing* and *nonincreasing*, respectively.

2. Preliminaries

Before proceeding to main theories we recall some related concepts and present one lemma that will be used in deriving the main results in the sequel.

For two random variables X and Y with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, density functions f and g, respectively, X is said to be smaller than Y in the

(i) *likelihood ratio order* (denoted as $X \le_{lr} Y$) if $\frac{g(t)}{f(t)}$ increases in $t \in \text{supp}(X) \cup \text{supp}(Y)$;
(ii) *hazard rate order* (denoted as $X \le_{hr} Y$) if $\frac{\bar{G}(t)}{\bar{F}(t)}$ increases in $t \in (-\infty, \max\{u_X, u_Y\})$;
(iii) *reversed hazard rate order* (denoted as $X \le_{rh} Y$) if $\frac{G(t)}{F(t)}$ increases in $t \in (\min\{l_X, l_Y\}, +\infty)$;
(iv) usual *stochastic order* (denoted as $X \le_{st} Y$) if $\bar{F}(t) \le \bar{G}(t)$ for all t;
(v) *convex order* (denoted as $X \le_{cx} Y$) if $\mathsf{E}[\phi(X)] \le \mathsf{E}[\phi(Y)]$ for any convex function ϕ, provided the expectations exist.

For more on stochastic orders one may refer to Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Li and Li (2013).

For any $1 \le i < j \le n$, denote the permutation

$$\tau_{ij}(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = (a_1, \ldots, a_j, \ldots, a_i, \ldots, a_n).$$

A multivariate real function $g(\boldsymbol{x})$ is said to be *arrangement increasing* (AI) if $g(\boldsymbol{x}) \ge g\left(\tau_{ij}(\boldsymbol{x})\right)$ for any $\boldsymbol{x} \in \mathbb{R}^n$ with $x_i \le x_j$ and $1 \le i < j \le n$. For any (i, j) such that $1 \le i < j \le n$, let $\Delta_{ij} g(\boldsymbol{x}) = g(\boldsymbol{x}) - g\left(\tau_{ij}(\boldsymbol{x})\right)$, and denote

$$\mathcal{G}_s^{ij}(n) = \left\{g(\boldsymbol{x}) : \Delta_{ij} g(\boldsymbol{x}) \ge 0 \quad \text{for any } x_j \ge x_i\right\},$$
$$\mathcal{G}_{rws}^{ij}(n) = \left\{g(\boldsymbol{x}) : \Delta_{ij} g(\boldsymbol{x}) \quad \text{is increasing in } x_j \text{ for } x_j \ge x_i\right\},$$
$$\mathcal{G}_{ws}^{ij}(n) = \left\{g(\boldsymbol{x}) : \Delta_{ij} g(\boldsymbol{x}) \quad \text{is increasing in } x_j\right\}.$$

According to Cai and Wei (2014) and Cai and Wei (2015), a random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is said to be