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On a class of dependent Sparre Andersen risk models and a bailout application

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ABSTRACT

In this paper a one-dimensional surplus process is considered with a certain Sparre Andersen type dependence structure under general interclaim times distribution and correlated phase-type claim sizes. The Laplace transform of the time to ruin under such a model is obtained as the solution of a fixed-point problem, under both the zero-delayed and the delayed cases. An efficient algorithm for solving the fixed-point problem is derived together with bounds that illustrate the quality of the approximation. A two-dimensional risk model is analyzed under a bailout type strategy with both fixed and variable costs and a dependence structure of the proposed type. Numerical examples and ideas for future research are presented at the end of the paper.

1. Introduction

For a given initial surplus $u \in \mathbb{R}_+$, we denote by $X = \{X(t), t \in \mathbb{R}_+\}$ the insurer's surplus, whose evolution at $t \in \mathbb{R}_+$ is given by

$$X(t) = u + ct - \sum_{i=1}^{N(t)} J_k.$$

Here, the premium rate c is assumed to be strictly positive. We denote by $N(t) = \max\{k \in \mathbb{N} : T_k \le t\}$ for $t \in \mathbb{R}_+$ the number of claims by time t and we assume independence among each generic pair interclaim time-claim size $\{(T_k, J_k)\}_{k=1}^{\infty}$. Furthermore, we assume that the surplus process $X(t)$ has a Sparre Andersen type dependence structure, defined by

$$P(T_k \in \mathrm{d}t, J_k \in \mathrm{d}x) = \alpha(\mathrm{d}t)\, \mathrm{e}^{Rx}\, \underline{r}\, \mathrm{d}x \quad t,\ x \in \mathbb{R}_+, \tag{1}$$

where $\alpha(\mathrm{d}t) \in \mathbb{R}^m$, is a $1 \times m$ distribution vector, R is an $m \times m$ sub-generator matrix, $\underline{r}$ an $m \times 1$ vector given by $\underline{r} = (-R)\underline{1}$, with $\underline{1}$ denoting the $m \times 1$ vector of ones. Note that within each pair interclaim time-claim size the random variables T_k and J_k are dependent, whereas the pairs $\{(T_k, J_k)\}_{k=1}^{\infty}$ are independent and identically distributed (i.i.d.) random variables. For this risk model, we assume that the safety loading condition for surplus $\{X_t,\ t \ge 0\}$ is satisfied, i.e. that $cE(T_1) > E(J_1)$.

In a similar fashion we introduce the corresponding delayed risk model where the joint distribution of the time of the first arrival and its jump-size differs from the joint distributions of time and jump-size corresponding to later arrivals, i.e. for $k = 1$

$$P(T_1 \in \mathrm{d}t, J_1 \in \mathrm{d}x) = \alpha_D(\mathrm{d}t)\mathrm{e}^{Rx}\, \underline{r}\, \mathrm{d}x, \tag{2}$$

while, for $k \ge 2$, $P(T_k \in \mathrm{d}t, J_k \in \mathrm{d}x)$ is given in (1).

Under the dependence structure assumed in (1) and (2) the claim sizes J_k are phase-type random variables with generator matrix described by R, whereas the vector $\alpha(\mathrm{d}t)$ (or $\alpha_D(\mathrm{d}t)$ in the delayed case) gives the joint density of the interclaim times and the phase of the claim $i \in 1, \ldots, m$ at the beginning of the claim. The generality of this dependence assumption comes from the fact that the distribution of the interclaim time T_k is equal to $\alpha(\mathrm{d}t)\underline{1}$, which may be defective, in the sense that $\int_0^{\infty} \alpha(\mathrm{d}t)\underline{1} < 1$. This is an important feature which will be illustrated by the applications described in the forthcoming sections.

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We remark that under the Markovian arrival process (MAP) assumption as in Badescu et al. (2005), the interclaim times and the claim sizes are conditionally independent phase-type random variables, given the initial phase of the claim. However, under this particular dependence structure the distribution of the interclaim times and the initial phase of the claim can be generally distributed and it is not restricted to a phase-type random variable. The phase-type renewal risk model studied in Asmussen and Albrecher (2010, Chapter 9), the MAP arrival risk model in Badescu et al. (2005) and the bivariate phase type renewal model from Badescu and Landriault (2009) are just particular cases that can be obtained from our underlying dependence assumption.

Willmot and Woo (2012) proposed a very general dependence structure of the form $\sum_{i=1}^{m}\sum_{j=1}^{n_i} k_{ij}(t)b_{ij}(x)$, where $k_{ij}(t)$ represents the interclaim probability density function (pdf) and $b_{ij}(x)$ the pdf of the claims. Note that when $n_i = 1$, $[\alpha(t)]_i = k_i(t)$ and $b_i(x) = [e^{Rx}\underline{r}]_i$ their formula reduces to (1). Despite of the generality of their model, Willmot and Woo (2012) obtain ruin related measures in an explicit numerically implementable form only when further assumptions, such as combination of Erlang type densities, are made on the interclaim and claims sizes.

The objective of our paper is two-fold. On the one hand, we are interested in deriving the Laplace transform of the time to ruin in such a general model, in a form that can be numerically implemented. On the other hand, by deploying the dependence structures proposed in (1) and (2) and the results obtained in the first part, we model and analyze certain scenarios that may be of interest in insurance, or in other areas like queueing theory. More specifically, in one of the applications that we consider in this paper we propose a two-dimensional surplus model that is driven by a bailout strategy. Such a bailout strategy is associated with the provision of financial support to a company or a country which faces serious financial difficulty or even bankruptcy. Along these lines, we propose a model with a main branch that injects capital into the subsidiary every time the surplus level of the subsidiary drops below a predefined level. Moreover, we assume that the main branch faces certain transaction costs associated to each of the capital injections. One of the questions of interest that we address in the paper is: How long does it take until the main branch will be bankrupted? Another useful application that can be tackled using the results derived in Sections 2 and 3 is a queueing model that involves priority queues with flushes, which will be briefly described in the last section which outlines some directions for future research.

The remainder of the paper is organized as follows. In Section 2 we obtain a fixed-point equation for the Laplace transform of the time to ruin in the zero-delayed and the delayed cases. The algorithm and the convergence of the proposed method are analyzed in Section 3 for the general dependence structures from (1) and (2). In Section 4 we study the proposed bailout strategy in an insurance context and we illustrate numerically the accuracy of our results. Section 5 presents conclusions and some further research applications.

2. The Laplace transform of the time to ruin

In this section we consider the one-dimensional risk model described in Section 1 focusing on the derivation of the Laplace transform of the time to ruin τ, defined as $\tau = \inf\{t \geq 0 : X(t) < 0\}$. For this, we let $\psi(t,u) = P_u(\tau < t)$ to be the finite time ruin probability, and denote its associated Laplace transform by $\hat{\psi}(q,u) = \int_0^\infty q e^{-qt}\psi(t,u)\,dt$. The results obtained in this section extend the results obtained under the renewal risk model by Asmussen and Albrecher (2010) in Chapter 9, Theorem 4.4, and the fixed-point problem in Proposition 4.3.

2.1. Dependent, non-delayed case

For the dependence structure given in (1) we recall that $\underline{r} := -R\underline{1}$ where $\underline{1}$ is a $\mathbb{R}_+^{m\times 1}$ column vector of ones. Furthermore, for two vectors v and w of same size we will use notation $v \leq w$ when each entry of vector v is less than the corresponding entry of w.

Definition 1. For any $m \times m$ negative-definite matrix Q and $1 \times m$ sub-probability vector-valued measure $\alpha(dt)$ on $\mathbb{R}_+ \setminus \{0\}$, we denote by $\hat{\alpha}(Q)$ the $1 \times m$ vector

$$\hat{\alpha}(Q) := \int_0^\infty \alpha(dt)e^{Qt}.$$

Let us now denote by $L_\alpha(q)$, $q \geq 0$, the Laplace transform of the (one-dimensional) sub-probability measure $\alpha(dt)\underline{1}$. One then has that L_α is linked to $\hat{\alpha}$ via

$$L_\alpha(q) := \int_0^\infty e^{-qt}\alpha(dt)\underline{1} = \hat{\alpha}(-qI)\underline{1}.$$

Theorem 1. *For any $q > 0$ the Laplace transform of the time to ruin is given by*

$$\hat{\psi}(q,u) = \hat{\rho}(q)e^{\Gamma(q)u}\underline{1}, \quad u \in \mathbb{R}_+, \tag{3}$$

where $\hat{\rho}(q)$ is a $1 \times m$ sub-probability vector satisfying the fixed-point equation

$$\hat{\rho}(q) = \hat{\alpha}(cR + c\,\underline{r}\,\hat{\rho}(q) - qI), \tag{4}$$

and $\Gamma(q) = R + \underline{r}\,\hat{\rho}(q)$.

If $q = 0$ there exists a $1 \times m$ sub-probability vector $\hat{\rho}(0)$ verifying (4) such that expression (3) holds for $\hat{\psi}(0,u)$.

Remark 1. The fixed point Eq. (4) is a matrix extension of the famous Kendall equation for the Laplace transform of the busy period density—see Feller (1971, XIV.4(4.1)). The connection to branching processes, which makes this equation transparent and explains its frequent appearance in applied probability, is also explained there. As commented by a referee, this type of equation can be easily "imported" from queueing theory by swapping the meaning of interclaim times and claim sizes, so that one gets a MAP with positive jumps (this is one form of queueing-risk duality).

Due to its appearance in three research fields, Eq. (4) has often been studied under different levels of generality—see for example Rogers (1994), Asmussen (1995), Pistorius (2006), Breuer (2008) and D'Auria et al. (2010).

Proving the uniqueness of the fixed point is not straightforward in the case $q = 0$, but see D'Auria et al. (2010). For completeness, we include our own proof of both cases in the Appendix.

Let us provide now a direct probabilistic argument for the fixed point equation. Starting at level u the process has to make a first-passage back to level u, this quantity being given by the LT of the busy period $\hat{\rho}(q)$ jointly with the phase of the claim at the down-crossing time. Furthermore, being in a claim phase, the surplus has to make a first passage to level 0. The associated Laplace transform of this first passage time together with the phase of the claim at the moment of crossing level 0 is given by $e^{\Gamma(q)u}$. The Laplace transform of the busy period $\hat{\rho}(q)$ is essential in the calculation of any ruin related measure and is a solution of the fixed-point Eq. (4). Badescu et al. (2005) showed that the equivalent busy period in the case of risk processes with Markovian claim arrivals and phase-type claim amounts is the unique solution of an equivalent matrix Riccati equation. By rephrasing the problem in terms of a Markov-additive process (MaP) the matrix $\Gamma(q)$ may be identified as the generator of the downward ladder process of a certain MaP satisfying the matrix equation $K(\Gamma(q)) = 0$, where $K(s) = -sI + R + \underline{r}\int_{(0,\infty)}\alpha(dt)e^{+cst-qt}$; if $q > 0$, this solution may be shown to be unique in the set of irreducible negative definite matrices.