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Move-based hedging of variable annuities: A semi-analytic approach

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ABSTRACT

In this paper, we propose a semi-analytic algorithm for measuring the mean and variance of the cost associated with a two-sided move-based hedging of options written on an underlying asset whose price follows a geometric Brownian motion. Numerical examples are presented to illustrate the computational accuracy and efficiency of the algorithm. We then apply the technique to a structured product-based variable annuity with buffered protection and an annual ratchet variable annuity.

1. Introduction

Variable annuities (VA) have been overshadowing traditional fixed annuities to become the leading form of protected investment worldwide. In the US, the total sales peaked in 2007, just before the financial crisis, at 184 billion dollars. The VA sales were 140 billion and 133 billion in 2014 and 2015, respectively. See LIMRA (2015) for more details. We also see large VA sales in other countries. For example, Towers Watson reports that UK variable annuity sales were 1.11 billion pounds and 0.87 billion pounds in 2013 and 2014, respectively. According to a report by Oliver Wyman Limited (2007), the popularity of these contracts is driven by the demographic changes taking place in most parts of the world. The over-50 population is getting larger, richer and more diverse in life styles. They not only demand access to market appreciation in order to keep abreast of the rising cost of living, but also expect protection for their assets and well-being given increased uncertainty in the volatility of asset returns. With the various investment options and multiple forms of guarantees variable annuities can offer, they are able to choose the best fit for their desired risk/return target.

The guaranteed minimum benefits in a VA contract generally fall into two categories, namely, the guaranteed minimum death benefits (GMDB) and the guaranteed minimum living benefits (GMLB). The latter can be further divided into three types, the guaranteed minimum income benefits (GMIB), the guaranteed minimum accumulation benefits (GMAB) and the guaranteed minimum withdrawal benefits (GMWB). In each case, the policyholder is promised some future payments regardless of the performance of the subaccounts. In the context of derivatives, this payoff feature is equivalent to that of a financial option and can therefore be hedged in a similar way.[1]

Broadly speaking, there are four approaches for VA hedging. The first is no hedging at all. For small VA blocks, running naked may be acceptable. However, for larger ones, it can be very risky due to high market volatility that has been exhibited in recent years. The second is to buy reinsurance or structured products from a third party. Such products may offer substantial protection to the insurer, but they can be expensive or even unavailable for at least two reasons: firstly, they are customized products designed particularly to meet the insurer's needs; secondly, reinsurers may be reluctant to offer coverage for the guaranteed variable annuities given the increased market risks of these products. The third

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[1] In this paper, we are only concerned about the financial risks of the VA product. The mortality risks are assumed to be diversifiable.

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choice is static hedging. This strategy aims to offset the embedded option in the VA contract through buying a portfolio of options from the market. Since the payoff structure of the embedded option is sometimes too exotic to be decomposed as a combination of the payoff of options available in the market, basis risk can be significant. Moreover, VA contracts usually span over a long period of time, but options longer than five years in maturity may not be available in the market for reasonable prices. The last strategy, which we will discuss in much detail here, dates back to the seminal option-pricing paper by Black and Scholes (1973), which proposed a continuous hedging strategy known as delta hedging. Since then, various refinements of delta hedging have been developed, including delta–gamma hedging, delta–vega hedging, delta–rho hedging, mean–variance hedging, local risk-minimization hedging, utility based hedging, etc. See Zakamouline (2009), Cerny (2007), and references therein for the comparison of alternative hedging strategies for different contract types, market conditions and model assumptions.

Though perfect in theory, continuous hedging demands rebalancing the hedging portfolio continuously in time, which is impossible in practice. Therefore, discrete hedging is employed as an approximation. Using this strategy, one constructs the same initial portfolio as continuous hedging, but adjusts it discretely in time. This difference gives rise to a non-self-financing replicating portfolio. Hence, the discrete hedging cost consists of two parts, one is the cost of constructing the initial hedging portfolio, the other is the cost associated with the subsequent rebalances.

There are mainly two kinds of discrete hedging strategies, time-based and move-based. The former hedges the option at equally spaced points in time. Boyle and Emanuel (1980) is one of the first studies on the distribution of the local tracking error of time-based discrete hedging. For the global tracking error, Bertsimas et al. (2000) derived the asymptotic distribution of the tracking error at each rebalancing point, as the number of rebalancing points tends to infinity; Hayashi and Mykland (2005) generalized the result of Bertsimas et al. (2000) to continuous Itô processes and also suggested a data-driven nonparametric hedging strategy for the case of unknown underlying dynamics; Angelini and Herzel (2009) computed the mean and the variance of the error of a hedging strategy for a contingent claim when trading in discrete time, which are valid for any fixed number of trading dates (however, their methods are not applicable to the move-based hedging. See the end of Section 2 of their paper for this point); Sepp (2012) derived an approximation for the probability density function of the profit-and-loss (PnL) of the time-based delta hedging strategy for vanilla options under the diffusion model and a proposed jump–diffusion model assuming discrete trading intervals and transaction costs. Despite its analytic tractability, the time-based strategy is a plain approximation to continuous hedging with no regard to the volatility risk. When the volatility is high, the value of the subaccount fluctuates more intensively over a short period, which necessarily requires frequent rebalancing of the hedging portfolio. A wiser choice for this situation is the move-based strategy, which hedges whenever the value of the underlying asset moves outside a prescribed region.

Cost estimation for the move-based discrete hedging is mathematically complex because it involves a number of dependent and right-censored hitting times as well as the values of the underlying asset at these hitting times. One approach in practice is to analyze the cost through Monte Carlo simulation. See Boyle and Hardy (1997) for example. Although straightforward by its nature, the Monte Carlo method has certain drawbacks. The path-dependency of the total hedging cost requires the generation of the whole trajectory of the asset price at each iteration, which is done by discretization. As pointed out in Glasserman (2003), this leads to bias in estimation, known as discretization error. Reducing the discretization error requires shortening the time step in path generation,[2] which turns out to be computationally time-consuming (see Section 3.1 for the comments on the computational time and the convergence rate of the Monte Carlo methods). Alternatively, one may use analytic approximation. For a certain type of move-based hedging, Dupire (2005) derived the limit of the end-of-period tracking error as the bandwidth goes to zero. Henrotte (1993) found approximation formulas for expected transactions costs and the variance of the total cash flow from both time-based and move-based strategies. Toft (1996) extended the work of Henrotte (1993) by showing how these expressions can be simplified and computed efficiently. As a matter of fact, all the analytic results we mentioned above are asymptotic. Indeed, Dupire's and Henrotte's expressions are obtained in the limit as the bandwidth, the transactions costs and the time between rebalancing points, respectively, go to zero. These limits, however, are clearly unrealistic. Hence, there is a need to develop a method that can estimate the cost directly.

In this paper, we investigate a move-based discrete hedging strategy and develop semi-analytic algorithms to compute the mean and variance of the rebalancing cost associated with the move-based strategy. Although quantifying the exact cost distribution seems infeasible, the calculation of the mean and variance is still of practical importance as they, respectively, are the proper measures of the expected return/loss and risk of the move-based hedging. We use a geometric Brownian motion model for the underlying asset and delta hedge with one hedging instrument only. One motivation for this particular choice of hedging strategy is that the execution of more advanced strategies requires hedging instruments other than the underlying subaccount. The selection of these instruments is company- and/or market-specific, due to issues such as liquidity, trading constraints and corporate policies. Hence, a universal analysis of more advanced hedging strategies is somewhat impossible.

The paper is organized as follows. Section 2 gives an overview of the move-based hedging strategy and the corresponding cost. Empirical analysis is conducted on the effectiveness of the move-based hedging. Section 3 describes the semi-analytic algorithm we develop for calculating the mean and variance of the hedging cost. Section 4 applies the technique to two types of variable annuities: the structured product-based VA and the annual ratchet VA. Section 5 concludes the paper.

2. A move-based hedging strategy

The main objective of this section is to provide a description of a specific move-based hedging strategy and to formulate the corresponding net hedging cost. In addition, we display some numerical examples in comparing the hedging effectiveness of the move-based and time-based strategies.

2.1. Description of move-based hedging and its cost

Suppose there is a risky asset paying dividends continuously at a proportional rate η. The time-t price S_t is modeled as a geometric Brownian motion given by

$$S_t = S_0 e^{(\mu-\eta-\sigma^2/2)t+\sigma W_t}, \quad t \geq 0, \tag{2.1}$$

[2] The discretization error is different from the statistical error. While the latter can be reduced by increasing the number of iterations, the former has to be reduced by shortening the time step in the process generation or by high order approximation methods. If we assume the geometric Brownian motion model for the underlying asset, the formula for the first order approximation of the process is the same as those with higher orders, so the only way to reduce discretization error is through finer time steps.