Insurance: Mathematics and Economics 71 (2016) 87–92

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal allocation of policy deductibles for exchangeable risks

Sirous Fathi Manesh [a], Baha-Eldin Khaledi [a,*], Jan Dhaene [b]

[a] Department of Statistics, Razi University, Kermanshah, Iran

[b] Department of AFI-Insurance, KU Leuven, Leuven, Belgium

ARTICLE INFO

Article history:
Received May 2015
Received in revised form
July 2016
Accepted 30 July 2016
Available online 1 September 2016

Keywords:
Hazard rate order
Increasing convex order
Likelihood ratio order
Log-concave density function
Majorization
Schur-concave function
Stochastic dominance

ABSTRACT

Let $X_1, \dots, X_n$ be a set of n continuous and non-negative random variables, with log-concave joint density function f, faced by a person who seeks for an optimal deductible coverage for these n risks. Let $\mathbf{d} = (d_1, \dots d_n)$ and $\mathbf{d}^* = (d_1^*, \dots d_n^*)$ be two vectors of deductibles such that $\mathbf{d}^*$ is majorized by $\mathbf{d}$. It is shown that $\sum_{i=1}^n (X_i \wedge d_i^*)$ is larger than $\sum_{i=1}^n (X_i \wedge d_i)$ in stochastic dominance, provided f is exchangeable. As a result, the vector $(\sum_{i=1}^n d_i, 0, \dots, 0)$ is an optimal allocation that maximizes the expected utility of the policyholder's wealth. It is proven that the same result remains to hold in some situations if we drop the assumption that f is log-concave.

1. Introduction

Consider an insurance agreement under which a policyholder is self-insured up to a pre-specified value, known as the deductible amount. The total loss X faced by the policyholder is a non-negative random variable, hereafter called a risk. If X exceeds the deductible amount d, the remaining risk, $X - d$, will be covered by the insurer, otherwise, X is covered by the policyholder himself. This type of insurance coverage is known as policy deductible (cf. Klugman et al., 2004). Under the deductible coverage, the risk X can be expressed as $X = (X \wedge d) + (X - d)_+$, where the first part is self-insured by the policyholder and the second part is indemnified by the insurer. Now consider a situation where the policyholder faces n risks $X_1, \dots, X_n$ which are insured under a policy deductible coverage. Suppose the amount d is the total deductible amount corresponding to all risks and the policyholder has the right to divide d into n non-negative values $d_1, \dots, d_n$ such that $\sum_{i=1}^n d_i = d$, and for $i = 1, \dots, n$, each d_i is the deductible corresponding to the risk X_i (cf. Cheung, 2007). The indemnified amount by the insurer is given by $\sum_{i=1}^n (X_i - d_i)_+$ and the retained risk which is not covered by the policy deductible coverage is given by $\sum_{i=1}^n (X_i \wedge d_i)$. With this set up, if w denotes the initial wealth after paying the required premium which is assumed not to depend on the choice of $(d_1, \dots, d_n)$, the policyholder's wealth is changed into $w - \sum_{i=1}^n (X_i \wedge d_i)$. It is of importance to the policyholder to obtain the optimal vector $\mathbf{d}'$ in the set

$$s_n(d) = \left\{ \mathbf{d} = (d_1, \dots, d_n) \in \mathbb{R}^{+n} \mid \sum_{i=1}^n d_i = d \right\}$$

for which the amount $w - \sum_{i=1}^n (X_i \wedge d_i)$ is maximized or equivalently, $\sum_{i=1}^n (X_i \wedge d_i)$ is minimized according to a given stochastic order criterion. Several optimization criteria (such as maximizing the expected utility, minimizing the variance, minimizing the probability of ruin, etc.) have been proposed, see for example Van Heerwaarden et al. (1989) or Denuit and Vermandele (1998). In this paper, following Cheung (2007), Hua and Cheung (2008a,b), Lu and Meng (2011), Xu and Hu (2012), You and Li (2014) and Hu and Wang (2014), we use the maximization of the expected utility criterion to find an optimal deductibles allocation. That is, we are looking for an allocation that maximizes

$$E\left[u\left(w - \sum_{i=1}^n (X_i \wedge d_i) \right) \right]$$

or, equivalently, minimizes

$$E\left[\tilde{u}\left(\sum_{i=1}^n (X_i \wedge d_i) \right) \right],$$

* Corresponding author.
E-mail addresses: sirus_60@yahoo.com (S.F. Manesh), bkhaledi@hotmail.com (B.-E. Khaledi), jan.dhaene@kuleuven.be (J. Dhaene).

http://dx.doi.org/10.1016/j.insmatheco.2016.07.010
0167-6687/

where $\tilde{u}(x) = -u(w-x)$ and u is a utility function which is assumed to be increasing (concave).

Let us recall the notion of majorization and various stochastic orderings which will be used to prove the main results in this paper.

Throughout this paper, we use increasing for non-decreasing and decreasing for non-increasing and assume that all the expectations of the random variables considered exist.

Let X and Y be univariate random variables with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, density functions f and g; hazard rates $r_F\ (= f/\bar{F})$ and $r_G\ (= g/\bar{G})$, respectively. Let l_X, l_Y and u_X, u_Y be the (finite or infinite) left and right endpoints of the support of X and Y, respectively. The random variable X is said to be smaller than random variable Y in the

- stochastic dominance order (denoted by $X \le_{st} Y$), if $E[\phi(X)] \le E[\phi(Y)]$ for all increasing function ϕ,
- increasing concave (convex) order (denoted by $X \le_{icv\,(icx)} Y$), if $E[\phi(X)] \le E[\phi(Y)]$ for all increasing concave (convex) function ϕ,
- hazard rate order (denoted by $X \le_{hr} Y$), if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$,
- likelihood ratio order (denoted by $X \le_{lr} Y$) if $g(x)/f(x)$ is increasing in $x \in (-\infty, \max(u_X, u_Y))$.

It is well known that $X \le_{st} Y$ is equivalent to $\bar{F}(x) \le \bar{G}(x)$ for all x. It is easy to see that $X \le_{hr} Y$, if and only if, for every x, $r_G(x) \le r_F(x)$. Note that we have the following chain of implications among the above stochastic orderings:

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{st} Y \Rightarrow X \le_{icv,icx} Y.$$

For more details on stochastic orders see e.g. Muller and Stoyan (2002), Denuit et al. (2005) or Shaked and Shanthikumar (2007).

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $(\tau_1, \ldots, \tau_n)$ be an arbitrary permutation of $(1, \ldots, n)$ and $x_{(i)}$ and $x_{[i]}$ denote the ith smallest and the ith largest of x_i's, respectively. The notion of majorization, which is one of the basic tools in establishing various inequalities in statistics and probability, is introduced next. For more details on majorization and its properties the reader is referred to Marshall et al. (2011).

Definition 1.1. A vector $\mathbf{x} \in \mathbb{R}^n$ is said to be majorized by another vector $\mathbf{y} \in \mathbb{R}^n$, notation $\mathbf{x} \le_m \mathbf{y}$, if $\sum_{i=1}^{j} x_{(i)} \ge \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Definition 1.2. A real valued function ϕ defined on set $\mathbb{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathbb{A}$, if $\phi(\mathbf{x}) \le (\ge) \phi(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ such that $\mathbf{x} \le_{\mathbf{m}} \mathbf{y}$.

The following lemma which has an important role in the proof of the main result of this paper is another version of result A.2.b on page 82 of Marshall et al. (2011) and they both have the similar proof.

Lemma 1.3. *Let $\mathbb{A}$ be a set with the property*

$$\mathbf{y} \in \mathbb{A} \quad \text{and} \quad \mathbf{x} \le_m \mathbf{y} \text{ implies } \mathbf{x} \in \mathbb{A}.$$

A continuous function ϕ defined on $\mathbb{A}$ is Schur-concave on $\mathbb{A}$ if and only if ϕ is symmetric and $\phi(x_1, s - x_1, x_3, \ldots, x_n)$ is increasing in $x_1 \le \frac{s}{2}$ for each fixed $s, x_3, \ldots, x_n$.

Next, we define log-concave functions.

Definition 1.4. A real valued function ϕ defined on set $\mathbb{A} = \{\mathbf{x} \in \mathbb{R}^n : \phi(\mathbf{x}) \ge 0\}$ is said to be log-concave, if for any $\mathbf{x}, \mathbf{y} \in \mathbb{A}$ and $\alpha \in [0, 1]$,

$$\phi(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \ge [\phi(\mathbf{x})]^{\alpha} [\phi(\mathbf{y})]^{1-\alpha}.$$

The class of log-concave probability density functions includes many common parametric families. For univariate, examples include normal densities, gamma densities with shape parameter greater than or equal to one, Weibull densities with exponents greater than or equal to one, beta densities with both parameters greater than or equal to one and logistic densities. For a more comprehensive list of univariate examples, see e.g. Bagnoli and Bergstrom (2005). Multivariate examples include the multivariate normal densities, Wishart densities and Dirichlet densities. Finally, we define exchangeable random variables.

Definition 1.5. The random variables $X_1, \ldots, X_n$ are said to be exchangeable, if the joint distribution function of $(X_{\tau_1}, \ldots, X_{\tau_n})$ is not dependent on $(\tau_1, \ldots, \tau_n)$.

For example, in the background risk models (BRM) it is common for random risks $X_1, \ldots, X_n$ to assume $X_i = h(Y_i, Z)$ for $i = 1, \ldots, n$ where $Y_1, \ldots, Y_n$ are stand-alone risks and Z is a background risk. In special cases, X_i can be defined as $X_i = ZY_i$ or $X_i = Y_i + Z$ for $i = 1, \ldots, n$. In BRM, if for every z, $Y_1|Z = z, \ldots, Y_n|Z = z$ are independent and identically distributed, then $X_1, \ldots, X_n$ will be exchangeable random risks. For more information about BRM see Pratt (1988), Finkelshtain et al. (1999), Tsanakas (2008), Franke et al. (2011) and Asimit et al. (2013). Furthermore, when analyzing real data it is of interest to have an arbitrary but given risks distribution, therefore the issue is to construct an exchangeable risks with given marginal distribution. Clearly the easiest but not the best case is that of independent and identically distributed random risks. However, the point here is, to allow a model having a possible dependence structure among the risks.
In the following, we assume that all considered random variables are absolutely continuous and non-negative.

Let $X_1, \ldots, X_n$ be a set of n risks faced by a policyholder and $\mathbf{d} = (d_1, \ldots, d_n)$ and $\mathbf{d}^* = (d_1^*, \ldots, d_n^*) \in S_n(d)$. Cheung (2007) proved that if either all X_i's are independent and $X_1 \le_{hr} \ldots \le_{hr} X_n$, or all X_i's are comonotonic and $X_1 \le_{st} \ldots \le_{st} X_n$, then

$$\sum_{i=1}^{n} (X_i \wedge d_{[i]}) \le_{icx} \sum_{i=1}^{n} (X_i \wedge d_{\tau_i}).$$

For more details on comonotonic random vectors, see e.g. Dhaene et al. (2002). Hua and Cheung (2008b) proved that for any random vector $(X_1, \ldots, X_n)$, we have that

$$\sum_{i=1}^{n} (X_i \wedge d_i) \le_{st} \left(\sum_{i=1}^{n} X_i\right) \wedge d,$$

which means that for any policyholder who has a deductible coverage for each risk with a fixed total deductible, the global insurance is the worst case. If the X_i's are independent with X_i, $i = 1, \ldots, n$, having log-concave density function, then Lu and Meng (2011) proved that if $X_1 \le_{lr} \ldots \le_{lr} X_n$, then

$$\mathbf{d} \ge_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^{n} (X_i \wedge d_{[i]}) \le_{st} \sum_{i=1}^{n} (X_i \wedge d_{[i]}^*). \tag{1.1}$$

Hu and Wang (2014) proved that (1.1) holds under the weaker condition that $X_1 \le_{hr} \ldots \le_{hr} X_n$.

In this paper, we drop the independence assumption of $X_1, \ldots, X_n$. In Theorem 2.4, we prove that under appropriate conditions, we have that

$$\mathbf{d} \ge_m \mathbf{d}^* \Longrightarrow \sum_{i=1}^{n} (X_i \wedge d_i) \le_{st} \sum_{i=1}^{n} (X_i \wedge d_i^*).$$

We show that the result holds in particular if the X_i's are exchangeable and the joint density function f is log-concave; see Theorem 2.6. The main consequence of this result (Corollary 2.7)