Insurance: Mathematics and Economics 71 (2016) 93–102

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Issues with the Smith–Wilson method

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ARTICLE INFO

Article history:
Received February 2016
Received in revised form
August 2016
Accepted 26 August 2016
Available online 6 September 2016

Keywords:
Smith–Wilson
Discount curve
Yield curve
Interpolation
Extrapolation
Hedging
Totally positive matrix
Solvency II

ABSTRACT

We analyse various features of the Smith–Wilson method used for discounting under the EU regulation Solvency II, with special attention to hedging. In particular, we show that all key rate duration hedges of liabilities beyond the Last Liquid Point will be peculiar. Moreover, we show that there is a connection between the occurrence of negative discount factors and singularities in the convergence criterion used to calibrate the model. The main tool used for analysing hedges is a novel stochastic representation of the Smith–Wilson method.

1. Introduction

In the present paper we analyse the mandated method for calculating the basic risk-free interest rate under Solvency II, the so-called Smith–Wilson method. This is an extra- and interpolation method, which is based on a curve fitting procedure applied to bond prices. The technique is described in a research note by Smith and Wilson from 2001, see Smith and Wilson (2000). Since Smith and Wilson (2000) is not publicly available, we have chosen to follow the notation of the European Insurance and Occupational Pensions Authority (EIOPA) given in EIOPA (2015). The primary aim with the current paper is to present problems with the Smith–Wilson method, especially with regards to hedging interest rate risk. We show analytically that the oscillating behaviour observed numerically by Ovtchinnikov (2015) and Rebel (2012) is always present (Section 3.1).

Our main theoretical tool is a representation of Smith–Wilson discount factors as expected values of a certain Gaussian process (Section 2).

With notation from EIOPA (2015), we have that the discount factor for tenor t, when fitted to N prices for zero coupon bonds with tenors $u_1, \dots, u_N$, is

$$P(t) := e^{-\omega t} + \sum_{j=1}^{N} \zeta_j W(t, u_j), \quad t \geq 0, \tag{1}$$

where $\omega := \log(1+\mathit{UFR})$ and *UFR* is the so-called Ultimate Forward Rate,

$$W(t, u_j) := e^{-\omega(t+u_j)} \left(\alpha \min(t, u_j) - e^{-\alpha \max(t,u_j)} \sinh(\alpha \min(t, u_j))\right), \tag{2}$$

and α is a parameter determining the rate of convergence to the *UFR*.

Based on the above it is seen that the ζ_j's are obtained by solving the linear equation system defined by (1) and (2) given by the specific time points $u_1, \dots, u_N$. Another name for u_N given in the regulatory framework is the Last Liquid Point (*LLP*), i.e. the last tenor of the supporting zero coupon bonds that are provided by the market.

The *UFR* is set to 4.2% for the Eurozone. In general, a higher value of α implies faster convergence to *UFR*. EIOPA (EIOPA, 2015, Paragraph 164) has decided that α should be set as small as possible, though with the lower bound 0.05, while ensuring that the forward intensity $f(t) := -\frac{d}{dt} \log P(t)$ differs at most 0.0001 from ω (defined above) at a certain tenor called the Convergence Point (*CP*):

$$|f(CP) - \omega| \leq 0.0001. \tag{3}$$

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http://dx.doi.org/10.1016/j.insmatheco.2016.08.009
0167-6687/

This optimisation of α can be troublesome to implement numerically since the left hand side of (3), seen as a function of α, can have singularities (Section 3.3).

We also point out below that having the forward yield to converge to a fixed *UFR* gives rise to an inconsistency with how the interest rate stress scenarios are specified in Solvency II (Section 3.4).

The method can also be applied to coupon bearing bonds, or swaps, but there is no loss in generality in considering only zero coupon bonds. The generalisation is particularly simple since a coupon bearing bond can be seen as a linear combination of zero coupon bonds and the Smith–Wilson method is linear in bond prices.

We also note that the market data that is used as input for the Smith–Wilson method should undergo a credit adjustment. This is nothing that we will specify further, but refer the reader to EIOPA (2015) and merely state that this adjustment is of no relevance for the results below. If anything, the variable credit adjustment will make hedging even harder. Note that the current market practice is to adjust for the credit risk by using two interest curves, one for discounting and one for projecting forward rates, see Ametrano and Bianchetti (2013) and Mercurio (2009).

Further, the methodology to derive the *UFR* is under review, see EIOPA (2016), and in future regulations one might expect *UFR* to change year by year.

For later convenience we will here state relevant abbreviations and notation:

- *LLP* is the Last Liquid Point for where the zero coupon bond market support ends.
- *UFR* is the Ultimate Forward Rate, i.e. 4.2% for most currencies.
- ω is the continuously compounded ultimate forward rate, i.e. $\omega = \log(1 + UFR)$.
- *CP* is the Convergence Point where the *UFR* should be reached.
- α is the mean reversion parameter that determines the rate of convergence to the *UFR*.
- $\boldsymbol{u}$ is a vector with tenors of the market zero coupon bonds.
- $\boldsymbol{p}$ is a vector of observed zero coupon prices at times to maturities $\boldsymbol{u}$, that is $\boldsymbol{p} = (p_1, \ldots, p_{LLP})'$.
- $\boldsymbol{r}$ is a vector of observed zero coupon spot rates at times to maturities $\boldsymbol{u}$, that is $\boldsymbol{r} = (r_1, \ldots, r_{LLP})'$, i.e. $\boldsymbol{p}_i := \boldsymbol{p}(\boldsymbol{r})_i = e^{-r_i u_i}$.
- $H(s,t)$ is the following function:

$$H(s,t) := \alpha \min(s,t) - e^{-\alpha \max(s,t)} \sinh(\alpha \min(s,t)).$$

- $W(s,t)$ is defined as $W(s,t) := e^{-\omega(s+t)} H(s,t)$.
- $\boldsymbol{Q}$ is a diagonal matrix with $\boldsymbol{Q}_{ii} = e^{-\omega u_i} =: \boldsymbol{q}_i, i = 1, \ldots, LLP$.
- $\boldsymbol{H}$ is a matrix with elements $\boldsymbol{H}_{ij} = (H(u_i, u_j))_{ij}$.
- $\boldsymbol{W}$ is a matrix with elements $\boldsymbol{W}_{ij} = (W(u_i, u_j))_{ij}$. Note that $\boldsymbol{W} = \boldsymbol{QHQ}$.
- $W(t, \boldsymbol{u})$ is defined as $W(t, \boldsymbol{u}) := (W(t, u_1), \ldots, W(t, u_{LLP}))'$. $H(t, \boldsymbol{u})$ is defined analogously.
- $\boldsymbol{b}$ is the solution to the equation $\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{QHQb}$ (note that this is the zero coupon case).
- $\sinh[\alpha \boldsymbol{u}']$ denotes $\sinh(\cdot)$ applied component-wise to the vector $\alpha \boldsymbol{u}'$.
- $P(t)$ is the discount function at t that suppress the dependence on the market support, i.e. $P(t) := P(t; \boldsymbol{p}(\boldsymbol{r})) \equiv P(t; \boldsymbol{p}) \equiv P(t; \boldsymbol{r})$.
- P^c is the present value of the cash flow $\boldsymbol{c}$ w.r.t. Smith–Wilson discounting using $P(t)$, i.e. $P^c := \sum_t c_t P(t)$. Hence, $P^c := P^c(t; \boldsymbol{p}(\boldsymbol{r})) \equiv P^c(t; \boldsymbol{p}) \equiv P^c(t; \boldsymbol{r})$.

2. Representing Smith–Wilson discount factors

The problems with the Smith–Wilson method that will be highlighted in later sections are centred around problems regarding hedging. In order to understand this in more detail we have found that the representation of the method from Andersson and Lindholm (2013) and Lagerås (2014) will prove useful. We will now give a full account of how the extrapolated discount factors of the Smith–Wilson method can be treated as an expected value of a certain stochastic process:

Let $\{X_t : t \geq 0\}$ be an Ornstein–Uhlenbeck process with $dX_t = -\alpha X_t dt + \alpha^{3/2} dB_t$, where $\alpha > 0$ is a mean reversion parameter, and $X_0 \sim N(0, \alpha^2)$ independent of B, and let $\bar{X}_t := \int_0^t X_s ds$ and $Y_t := e^{-\omega t}(1 + \bar{X}_t)$. Given this we can state the following theorem:

Theorem 1. $P(t) = \mathbb{E}[Y_t | Y_{u_i} = p_{u_i}; i = 1, \ldots, N]$.

In other words: the Smith–Wilson bond price function can be interpreted as the conditional expected value of a certain *non-stationary* Gaussian process. Note that α will govern *both* mean reversion and volatility.

Since $\{Y_t : t \geq 0\}$ is a Gaussian process we have that $P(t)$, being a conditional expected value, is an affine function of $\boldsymbol{p}$:

$$\begin{aligned} P(t) &= \mathbb{E}[Y_t] + \mathrm{Cov}[Y_t, \boldsymbol{Y}]\mathrm{Cov}[\boldsymbol{Y}, \boldsymbol{Y}]^{-1}(\boldsymbol{p} - \mathbb{E}[\boldsymbol{Y}]) \\ &=: \beta_0(t) + \boldsymbol{\beta}(t)'\boldsymbol{p}, \end{aligned} \quad (4)$$

where $\beta_0(t)$ and $\boldsymbol{\beta}(t)$ are functions of t, but not $\boldsymbol{p}$, if α is considered a fixed parameter. If α is set by the convergence criterion, β_0 and $\boldsymbol{\beta}$ are functions of $\boldsymbol{p}$ as well as t.

The main aim of this paper is to analytically show problems inherent in the Smith–Wilson method which will affect hedging of liabilities. From this perspective it is evident that the reformulation of the bond price function according to Eq. (4) will prove useful, and in particular the behaviour of the β's will be of interest:

Theorem 2. *If* $t > u_N$, $\mathrm{sign}(\beta_i(t)) = (-1)^{N-i}$ *for* $i = 1, \ldots, N$.

This has peculiar consequences for hedging interest rate risk. The proofs of Theorems 1 and 2 are given in Section 5.

3. Problems with the Smith–Wilson method

When the Smith–Wilson method was proposed by the regulator, it was known to have both advantages and disadvantages, see EIOPA (2010). One of the advantages expressed at that time is that it is not only an interpolation method, but also an extrapolation method.

Hagan and West (2006) have produced a broad survey of interpolation methods where they list several desiderata. A good interpolation method should

(a) match market data and be computationally fast to fit,
(b) generate smooth forward rates,
(c) generate non-negative forward rates,
(d) generate stable forward rates, i.e. small changes in market input should not generate large changes in forward rates,
(e) generate hedges that are local, i.e. a liability should be hedged with market instruments with tenors as close as possible to that of the liability.

Let us elaborate on these, especially in the context of valuing and hedging insurance liabilities which requires extrapolation as well as interpolation, and check how well the Smith–Wilson method fulfils the criterion.

The first part of Point (a), concerning match to market data, is required since a liability with a cash flow identical to one of a traded financial instrument should have a value equal to that instrument, and it can be hedged by buying that instrument. The Smith–Wilson method satisfies this condition. A computationally fast fit is necessary if one wants to perform Monte Carlo simulations, and even though the Smith–Wilson method only has