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Tail asymptotics of generalized deflated risks with insurance applications☆

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ABSTRACT

Let X and $S \in (0, 1)$ be two independent risk variables. This paper investigates approximations of generalized deflated risks $\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}$ with a flexible constant $\kappa \geq 0$ under extreme value theory framework. Our findings are illustrated by three applications concerning higher-order tail approximations of deflated risks as well as approximations of the Haezendonck–Goovaerts and expectile risk measures. Numerical analyses show that higher-order approximations obtained in this paper significantly improve lower-order approximations.

1. Introduction

Throughout this paper, let X be a risk variable and $S \in (0, 1)$ be another risk variable independent of X. Of interest is the following quantity of generalized deflated risks

$$\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}, \tag{1.1}$$

where κ is a non-negative flexible constant such that $\mathbb{E}\{|X|^\kappa\} < \infty$ and $\mathbb{I}\{\cdot\}$ stands for the indicator function. In various theoretical and practical situations, a natural question is how the approximation of (1.1) is influenced by the tail behavior of the principal risk X and the finance risk S; the recent contributions (Hashorva et al., 2014, 2010) investigated the issue for the deflated risks SX, i.e., $\kappa = 0$, concerned with time-values of capital in insurance claims. Another motivation for considering approximations of (1.1) comes from finance and risk management fields. Specifically, for $S \sim Beta(1, \kappa)$ a Beta distributed risk variable, independent of risk variable X with $\mathbb{E}\{|X|^\kappa\} < \infty$, (1.1) is reduced to $\mathbb{E}\left\{(X - x)_+^\kappa\right\}$ with $x_+ = \max(x, 0)$, which is involved in risk measures such as the Haezendonck–Goovaerts (H–G) and the expectile risk measures; see e.g., Bellini and Bernardino (2015), Bellini et al. (2014), Mao and Hu (2012), Mao et al. (2015) and Tang and Yang (2012). Additionally, the quantity in (1.1) is closely related to Weyl fractional-order integrals and random products in the log-linear models; see e.g., Hashorva and Pakes (2010), Janßen and Drees (2015), Pakes and Navarro (2007) for related discussions.

This paper aims to study approximations of (1.1). We remark that one can consider generally $\mathbb{E}\{\mathcal{L}(X)\mathbb{I}\{SX > x\}\}$ with some regular variation functions $\mathcal{L}(\cdot)$. The main contributions of this paper concern second- and third-order approximations of (1.1) which are illustrated by several examples and numerical analyses. The main methodology is based on higher-order regular variation theory in de Haan and Stadtmüller (1996), Fraga Alves et al. (2006) and Wang and Cheng (2006). As mentioned above for potential applications, we first obtain tail asymptotics of deflated risks in Theorems 3.1 and 3.2, refining those in Hashorva et al. (2014), Hashorva et al. (2010), and then, with the aid of Corollary 2.5, we

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investigate approximations of the expectile and H–G risk measures which were initially studied by Bellini and Bernardino (2015) and Tang and Yang (2012), respectively. These two competitive risk measures have received increasing attention; see e.g., Bellini et al. (2014), Cai and Weng (2016) and Tang and Yang (2014). As expected, Theorems 3.4 and 3.5 are, respectively, significant refinements of findings displayed in Mao and Hu (2012), Tang and Yang (2012) and Bellini and Bernardino (2015), Mao et al. (2015). As a consequence, our findings enrich greatly related discussions in both actuarial science and statistical practice.

We organize this paper as follows. In Section 2, we display our main results. Section 3 is devoted to the applications. Several examples and numerical analyses are given in Section 4 to illustrate our findings. The proofs are relegated to Section 5.

2. Main results

We start with the definitions and some properties of regular variations which are crucial to establish main results. A measurable function $f : [0, \infty) \to \mathbb{R}$ is said to be an extended regularly varying function (ERV) at infinity with index $\gamma \in \mathbb{R}$, denoted by $f \in \mathrm{ERV}_\gamma$, if (cf. Bingham et al., 1987, de Haan and Ferreira, 2006)

$$\lim_{t\to\infty} \frac{f(tx)-f(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} =: D_\gamma(x) \tag{2.1}$$

holds for all $x > 0$ and an eventually positive function $a(\cdot)$, which is referred to as the auxiliary function. We mainly employ higher-order regular variations refining (2.1) to study approximations of (1.1). Specifically, f is the so-called second-order extended regular variation with parameters $\gamma \in \mathbb{R}$ and $\rho \le 0$, denoted by $f \in 2\mathrm{ERV}_{\gamma,\rho}$, if there exists further some auxiliary function $A(\cdot)$ with constant sign near infinity satisfying $\lim_{t\to\infty} A(t) = 0$, such that for all $x > 0$ (cf. de Haan and Stadtmüller, 1996, Resnick, 2007)

$$\lim_{t\to\infty} \frac{(f(tx)-f(t))/a(t) - D_\gamma(x)}{A(t)} = \int_1^x y^{\gamma-1}\int_1^y u^{\rho-1}\,du\,dy =: H_{\gamma,\rho}(x). \tag{2.2}$$

Furthermore, we write $f \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$ meaning that f is a third-order regular variation with parameters $\gamma \in \mathbb{R}$ and $\rho, \eta \le 0$, if there exists a third-order auxiliary function $B(\cdot)$ with constant sign near infinity satisfying $\lim_{t\to\infty} B(t) = 0$, such that for all $x > 0$ (cf. Fraga Alves et al., 2006, Wang and Cheng, 2006)

$$\lim_{t\to\infty} \frac{\frac{(f(tx)-f(t))/a(t)-D_\gamma(x)}{A(t)} - H_{\gamma,\rho}(x)}{B(t)} = \int_1^x y^{\gamma-1}\int_1^y u^{\rho-1}\int_1^u v^{\eta-1}\,dv\,du\,dy =: R_{\gamma,\rho,\eta}(x). \tag{2.3}$$

As is well-known, f is the so-called first-order, second-order and third-order regular variation, if (2.1), (2.2) and (2.3) hold with $(f(tx)-f(t))/a(t)$ replaced by $f(tx)/f(t)$, and $D_\gamma(x)$, $H_{\gamma,\rho}(x)$ and $R_{\gamma,\rho,\eta}(x)$ by x^γ, $x^\gamma D_\rho(x)$ and $x^\gamma D_{\rho+\eta}(x)$, respectively, denoted by $f \in \mathrm{RV}_\gamma$, $f \in 2\mathrm{RV}_{\gamma,\rho}$ and $f \in 3\mathrm{RV}_{\gamma,\rho,\eta}$ accordingly. Note in passing that for $f \in 3\mathrm{RV}_{\gamma,\rho,\eta}$ with auxiliary functions $A(\cdot)$ and $B(\cdot)$, it follows from Fraga Alves et al. (2006) that $A \in 2\mathrm{RV}_{\rho,\eta}$ with auxiliary function $B(\cdot)$, and $|B| \in \mathrm{RV}_\eta$. We refer to Cai et al. (2013), Li et al. (2011), Mao and Hu (2012) for more applications of ERV and RV in extreme value statistics.

Throughout this paper, we keep the notation in the introduction and set further $\overline{Q} = 1 - Q$ for some function Q and $U(t) = F^{\leftarrow}(1-1/t)$ for the tail quantile function of $X \sim F$, and $S \sim G$. All limits are taken as the argument goes to the right endpoint of X (denoted by $x_F := U(\infty)$) unless otherwise stated.

We consider first $\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}$ with X having heavy tails. Denote for $\alpha > \kappa \ge 0$, $\varrho, \varsigma \le 0$

$$\begin{cases} d_{0,\kappa} = \mathbb{E}\left\{S^{\alpha-\kappa}\right\}, \\ d_{1,\kappa} = \dfrac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa}\right\}}{\varrho} + \dfrac{\kappa\mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\}}{\alpha(\alpha-\kappa-\varrho)}, \\ d_{2,\kappa} = \dfrac{\kappa\left(\mathbb{E}\left\{S^{\alpha-\kappa-2\varrho}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\}\right)}{\alpha\varrho(\alpha-\kappa-\varrho)} \\ d_{3,\kappa} = \dfrac{\kappa}{\alpha}\left(\dfrac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa-\varrho}\right\}}{(\alpha-\kappa-\varrho)\varsigma} + \dfrac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\}}{\alpha-\kappa-\varrho-\varsigma}\right) \\ \qquad + \dfrac{\mathbb{E}\left\{S^{\alpha-\kappa-\varrho-\varsigma}\right\} - \mathbb{E}\left\{S^{\alpha-\kappa}\right\}}{\varrho+\varsigma}. \end{cases} \tag{2.4}$$

Theorem 2.1. *If $\overline{F} \in 2\mathrm{RV}_{-\alpha,\varrho}$, $\alpha > \kappa$, $\varrho \le 0$ with auxiliary function $A(\cdot)$, then*

$$\frac{\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}}{x^\kappa \overline{F}(x)} = \frac{\alpha}{\alpha-\kappa}\big(d_{0,\kappa} + d_{1,\kappa}A(x)(1+o(1))\big). \tag{2.5}$$

If further $\overline{F} \in 3\mathrm{RV}_{-\alpha,\varrho,\varsigma}$, $\alpha > \kappa$, $\varrho, \varsigma \le 0$ with auxiliary functions $A(\cdot)$ and $B(\cdot)$, then

$$\frac{\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}}{x^\kappa \overline{F}(x)} = \frac{\alpha}{\alpha-\kappa}\Big(d_{0,\kappa} + A(x)\Big(d_{1,\kappa} + \big(d_{2,\kappa}A(x) + d_{3,\kappa}B(x)\big)(1+o(1))\Big)\Big). \tag{2.6}$$

Here $d_{i,\kappa}$, $0 \le i \le 3$ are given in (2.4).

Remark 2.2. It follows that, $\int_x^\infty y^c(y-x)^\beta dF(y) = \mathbb{E}\{X^{\beta+c}\mathbb{I}\{SX > x\}\}$ with $S \sim Beta(1, \beta)$ and $\mathbb{E}\left\{|X|^{\beta+c}\right\} < \infty$. Therefore, Theorem 2.1 extends Theorem 7.2 in Hashorva and Pakes (2010) concerning the Weyl fractional-order integrals.

Next, we consider X with light tails. Set below for $\gamma \in \mathbb{R}$, $\alpha > 0$, $\rho, \eta \le 0$, $\varrho < 0$, and D_γ, $H_{\gamma,\rho}$, $R_{\gamma,\rho,\eta}$ given by (2.3)

$$\begin{cases} L_\alpha = \displaystyle\int_0^1 (D_\gamma(1/s))^\alpha\,ds, \\ M_{\alpha,l} = c_{\alpha,l}\displaystyle\int_0^1 (D_\gamma(1/s))^{\alpha-l}(H_{\gamma,\rho}(1/s))^l\,ds \\ N_{\alpha,l,\varrho} = c_{\alpha,l}\displaystyle\int_0^1 (D_\gamma(1/s))^{\alpha-l}(H_{\gamma,\rho}(1/s))^l \\ \qquad \times \dfrac{(D_\gamma(1/s))^{-\varrho}-1}{\varrho}\,ds \\ Q_\alpha = \alpha\displaystyle\int_0^1 (D_\gamma(1/s))^{\alpha-1}R_{\gamma,\rho,\eta}(1/s)\,ds, \\ c_{\alpha,l} = \displaystyle\prod_{i=0}^{l-1}(\alpha-i)/l!, \quad l \in \mathbb{N}. \end{cases} \tag{2.7}$$

Theorem 2.3. *If $U \in 2\mathrm{ERV}_{\gamma,\rho}$, $\gamma, \rho \le 0$ with auxiliary functions $a(\cdot)$, $A(\cdot)$, and $\overline{G}(1-1/x) \in 2\mathrm{RV}_{-\alpha,\varrho}$, $\alpha > 0$, $\varrho < 0$ with auxiliary function $\widetilde{A}(\cdot)$, then with $\varphi_t = U(t)/a(t)$, $t = 1/\overline{F}(x)$*

$$\frac{\mathbb{E}\{X^\kappa \mathbb{I}\{SX > x\}\}}{x^\kappa \overline{F}(x)\overline{G}(1-1/\varphi_t)} = L_\alpha + \left(M_{\alpha,1}A(t) + N_{\alpha,0,\varrho}\widetilde{A}(\varphi_t) + (\kappa-\alpha)\frac{L_{\alpha+1}}{\varphi_t}\right)(1+o(1)). \tag{2.8}$$

If further $U \in 3\mathrm{ERV}_{\gamma,\rho,\eta}$, $\gamma, \rho, \eta \le 0$ with auxiliary functions $a(\cdot)$, $A(\cdot)$ and $B(\cdot)$, and $\overline{G}(1-1/x) \in 3\mathrm{RV}_{-\alpha,\varrho,\varsigma}$, $\alpha > 0$, $\varrho, \varsigma < 0$