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Time-consistent actuarial valuations[☆]

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ABSTRACT

Time-consistent valuations (i.e. pricing operators) can be created by backward iteration of one-period valuations. In this paper we investigate the continuous-time limits of well-known actuarial premium principles when such backward iteration procedures are applied. This method is applied to an insurance risk process in the form of a diffusion process and a jump process in order to capture the heavy tailed nature of insurance liabilities. We show that in the case of the diffusion process, the one-period time-consistent Variance premium principle converges to the non-linear exponential indifference price. Furthermore, we show that the Standard-Deviation and the Cost-of-Capital principle converge to the same price limit. Adding the jump risk gives a more realistic picture of the price. Furthermore, we no longer observe that the different premium principles converge to the same limit since each principle reflects the effect of the jump differently. In the Cost-of-Capital principle, in particular the VaR operator fails to capture the jump risk for small jump probabilities, and the time-consistent price depends on the distribution of the premium jump.

1. Introduction

Standard actuarial premium principles usually consider a static premium calculation problem: what is today's price of an insurance contract with payoff at time T. Textbooks such as those by Bühlmann (1970), Gerber (1979), and Kaas et al. (2008) provide examples of this. The study of risk measures and the closely related concept of monetary risk measures have also been studied in static settings by authors such as Artzner et al. (1999) and Cheridito et al. (2005). The study of utility indifference valuations has mainly confined itself to static settings as well. Different applications can be found in papers by Young and Zariphopoulou (2002), Henderson (2002), Hobson (2004), Musiela and Zariphopoulou (2004) and Monoyios (2006), and the book by Carmona (2009).

Financial pricing usually considers a "dynamic" pricing problem, and looks at how the price evolves over time until the final payoff date T. This dynamic perspective is driven by the focus on hedging and replication. The literature was started by the seminal paper of Black and Scholes (1973) and has been immensely generalized to broad classes of securities and stochastic processes; see Delbaen and Schachermayer (1994). Some researches in the last two decades focus on combining actuarial and financial pricing. See for example, Wang (2002) where he used distortion risk measures to price both types of risks and Goovaerts and Laeven (2008) where they used actuarial risk measures to price financial derivatives.

In recent years, researchers have begun to investigate risk measures in a dynamic setting, where the question of constructing time-consistent (or "dynamic") risk measures has been investigated. See Riedel (2004), Cheridito et al. (2006), Roorda et al. (2005), Rosazza Gianin (2006), and Artzner et al. (2007). As an example, Stadje (2010) showed how a large class of dynamic convex risk measures in continuous-time can be derived from the limit of their discrete time versions. Moreover, Jobert and Rogers (2008) showed how time-consistent valuations can be constructed through the backward induction of static one-period risk measures (or "valuations"). And later, Pelsser and Stadje (2014) studied time and market consistency of the well-known actuarial principles in a dynamic setting by using a two-step valuation method.

Insurance risk can be modeled in a stochastic way by using a diffusion process. However, it is usual that insurance risks exhibit jump type movements in their evolution, and the data usually contain a number of extreme events and stylized facts

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usually exist such as fat-tailed and skewed distributions. This justifies the usage of a jump component to draw a realistic inference about the dynamic pricing framework. Merton (1976) introduced the jump–diffusion model to price options by assuming discontinuity in returns. The model was developed extensively for financial modeling, actuarial valuation and the pricing of different derivatives and contingent claims in incomplete markets. There are numerous works about the jump process in finance; see for example Cont and Tankov (2012). For an introduction to the application of diffusion and jump processes in insurance see, for example, Korn et al. (2010) and for more specific actuarial applications see Biffis (2005), Verrall and Wüthrich (2012), Chen and Cox (2009), and Jang (2007). Some researchers have generalized the concept of time-consistent dynamic risk measures by using jump–diffusion processes when underlying risks include jumps. See for example Bion-Nadal (2008). The idea was developed in actuarial valuation using Backward Stochastic Differential Equations (BSDE) and g-expectations as more powerful tools to deal with non-linear pricing operators such as different premium principles. There are also a number of studies about modeling jumps with BSDEs in valuation and portfolio choice. See for example the textbook by Delong (2013) and the paper by Laeven and Stadje (2014).

In this paper we investigate well-known actuarial premium principles such as the Variance principle and the Standard-Deviation principle, and we study their time-consistent extension. We first consider one-period valuations, then extend this to a multi-period setting using the backward iteration method of Jobert and Rogers (2008) for a given discrete time-step $(t, t+\Delta t)$, and finally consider the continuous-time limit for $\Delta t \to 0$. A more general setting to model the insurance risk could be "infinite activity Lévy process" where it allows for infinite number of jumps for any finite time interval. However, as it does not seem realistic for an insurance process to have infinite number of jumps when $(t, t+\Delta t)$ is infinitesimally small, we waive the infinite activity Lévy process and we focus on investigating the method with simple diffusion and jump–diffusion processes.

We apply backward iteration to a simple diffusion model to show that the one-period Variance premium principle converges to the non-linear exponential indifference valuation. Furthermore, we study the continuous-time limit of the one-period Standard-Deviation principle and the Cost-of-Capital principle, and establish that in the diffusion setting, they converge to the same limit represented by an expectation under an equivalent martingale measure. We apply the same approach to the jump–diffusion setting and show that the time-consistent prices for different premium principles in the limit converge to different results than in the diffusion case. We mainly used the infinitesimal generator together with Itô's formula for different forms of the premium with the underlying process $y(t)$ in both diffusion and jump–diffusion models. See for example the book by Shreve (2010) about martingales and Itô's formula and the book by Øksendal (2003) for infinitesimal generators. As an exception, in the Cost-of-Capital principle under the jump setting, we have to make inference about the distribution of the insurance process under VaR operator. To do so, we will assume the jump process as a special case of the Lévy process and find its characteristic function. To get more insight about the Lévy process and its applications, see for example Figueroa-López (2012) and the textbook by Barndorff-Nielsen et al. (2001). We apply this method to a health process to price a stylized life insurance product and we use a Markov chain approximation to discretize the time and state space of the underlying insurance process. See for example Kushner and Dupuis (2001), Duan et al. (2003), and Tang and Li (2007) for the idea of using a Markov chain approximation to price contingent payoffs in theory and application.

The rest of this paper is organized as follows. In Section 2 we define the time-consistent valuation operators and explain about the backward iteration method used to construct it. In Section 3 we derive the time-consistent extension of the Variance premium principle with and without discounting. Section 3 also includes a benchmark version of this premium and the Mean Value principle as a more general pricing rule. In Section 4, we derive the time-consistent value of the Standard-Deviation and Cost-of-Capital premium principles. In both sections, we assume that the underlying pure insurance risks follow a diffusion process and we represent the results by means of the related Partial Differential Equation (PDE). In Section 5, we assume that the underlying process includes a Poisson jump component and we derive the time-consistent value for the principles (that we used in Sections 3 and 4) in the form of the Partial Integro-Differential Equations (PIDEs). In Section 6, we provide an example of the pricing procedure for a stylized insurance product using the Markov chain method and show the convergence of the numerical algorithm to analytical solution. We summarize and conclude in Section 7.

2. Time-consistent valuation operators

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the underlying probability space and $X(\omega)$ and $Y(\omega)$ be the stochastic insurance risk processes defined over the σ-algebra $\mathscr{F}$. Indexing for the time $0 \leq t \leq T$, we form the filtration $\mathscr{F}_t$ as the collection of the σ-algebras. In this paper, we limit ourselves to the square integrable functions and denote the space of such random variables as $\mathscr{L}^2(\Omega, \mathscr{F}_t, \mathbb{P})$.

Time consistency postulates that the order of riskiness of different portfolios measured by a dynamic risk measure in the future time is consistent with their riskiness at any time prior to that point in time and remains the same. It suggests that if at any time t the position A forms a higher risk than position B, the level of risk will be higher for all $s < t$. The next definition formulates the time consistency of a risk measure.

Definition 2.1. A dynamic risk measure (ρ_t) is Time-Consistent if and only if, for all $0 \leq t \leq T$ and $\forall X, Y \in L^2(\mathscr{F}_t)$,

$$\rho_T(X) \leq \rho_T(Y)\ P\text{-a.s.} \Rightarrow \rho_t(X) \leq \rho_t(Y)\ P\text{-a.s.} \tag{2.1}$$

or equivalently by its "recursive" form for $\forall s = \Delta t, 2\Delta t, \dots, T-t$, we have $\rho_t = \rho_t(-\rho_{t+s})$,

where $\rho_t : \mathscr{L}^2(\mathscr{F}_T) \to \mathscr{L}^2(\mathscr{F}_t)$ is a conditional risk measure for all $T \geq t$. The definition for non-negative risks (e.g. insurance losses) then becomes,

$$\rho_t = \rho_t(\rho_{t+s}). \tag{2.2}$$

Similar notions of time consistency can be found in Föllmer and Penner (2006), Cheridito and Stadje (2009), and Acciaio and Penner (2011).

We construct the time-consistent valuation operators for the insurance risks by the recursive form (2.2) and we use the backward induction method introduced by Jobert and Rogers (2008). In general we assume that the insurance process evolves during the time period $[0, T]$ and that at maturity time T it falls into a bounded state space where we can also define the state space of the contingent payoff. Based on this method, time consistency can be achieved for the price operator by decomposing the valuation operator into a family of one-period pricing operators that can only be valuated in shorter intermediate time periods.

To derive the time-consistent actuarial value at the present time $t = 0$, we divide the valuation period $[0, T]$ into a discrete set $\{0, \Delta t, 2\Delta t, \dots, T-\Delta t, T\}$ so that we can perform a multi-period valuation by applying the one-period pricing operator to all subintervals denoted by $(t, t+\Delta t)$. We use well-known actuarial premium principles such as the Variance, Standard-Deviation and Cost-of-Capital principles as pricing operators. Our aim is to apply the backward iteration method to all subintervals $(t, t+\Delta t)$