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Love and death: A Freund model with frailty

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ABSTRACT

We introduce new models for analyzing the mortality dependence between individuals in a couple. The mortality risk dependence is usually taken into account in the actuarial literature by introducing special copulas with continuous density. This practice implies symmetric effects on the remaining lifetime of the surviving spouse. The new model allows for both asymmetric reactions by means of a Freund model, and risk dependence by means of an unobservable common risk factor (or frailty). These models allow for distinguishing in the lifetime dependence the component due to common lifetime (frailty) from the jump in mortality intensity upon death of spouse (Freund model). The model is applied to the pricing of insurance products such as joint life policy, last survivor insurance, or contracts with reversionary annuities. A discussion of identification is also provided.

1. Introduction

This paper introduces new models for analyzing the mortality dependence between individuals in a couple. This type of model is needed for risk management and pricing of life insurance products written on two lives, such as joint life policy, last survivor insurance policy, or contract with reversionary annuities.

The basic actuarial literature usually assumed the independence between the spouses' mortality risks. Recently the mortality risk dependence has been introduced by means of copulas (see e.g. Frees et al., 1996; Youn and Shemyakin, 1999; Carriere, 2000; Denuit et al., 2001; Shemyakin and Youn, 2006; Luciano et al., 2008, 2010), and the effect of this dependence on the risk premia starts to be measured. However, standard copula models assume continuous copula densities. This implies symmetric reactions of the mortality of a member of the couple when the other dies. An alternative consists in introducing jumps in mortality intensity (the Freund model) at the time of death of the spouse, to capture the death of a spouse (see e.g. Spreeuw and Wang, 2008; Ji et al., 2011; Spreeuw and Owadally, 2013). Our paper extends this literature by mixing the Freund model, which allows for asymmetric reactions of the mortality intensities at a death event, with unobservable common factor (or frailty), which underlies many usual Archimedean copulas.[1]

The basic Freund model and its properties in terms of conditional intensities are presented in Section 2. This model allows for jump in the mortality intensity of a given spouse when the other spouse dies. The magnitude of this jump and its variation with respect to the age of the couple is the basis for constructing a convenient association measure, useful to analyze the broken-heart syndrome. The Freund model is extended in Section 3 to include common unobserved static frailty. In particular we discuss the properties of Freund models with latent intensities which are exponential affine functions of the frailty. These models are used in Section 4 to derive the prices of various contracts written on two lives. We consider these prices at the inception of the contract as well as during its lifetime. We emphasize the effect of the dependence between the mortality risks of the two spouses on these prices. Section 5 concludes. Proofs are gathered in appendices and a discussion on the identification issues is provided in Appendix D.

2. The basic Freund model

This type of model has been introduced by Freund (1961) to construct bivariate survival models for dependent duration

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[1] More precisely Archimedean copulas with completely monotone generators (see McNeil and Nešlehová, 2009).

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variables, while still featuring the lack of memory property. It has been noted by Tosch and Holmes (1980) that such models have an interpretation in terms of latent variables. We follow this interpretation. The model is written for a given couple, without specifying the index of the couple and possibly its observed characteristics such as the birth dates of the spouses, the difference between their ages (Youn and Shemyakin, 1999), or their age at the time of their marriage or common law relationship. In the application, such static couple characteristics will be introduced to capture the generation effects. The analysis is in continuous time and the lifetime variables are continuous variables.

2.1. The latent model

Let us consider a given couple with two spouses 1 and 2. The potential lifetimes of individuals 1 and 2, when both are alive, are denoted by X_1 and X_2, respectively. To get a unique time origin for the two members of the couple, these latent lifetimes are measured since the beginning of the common life. A first individual in the couple dies at date $\min(X_1, X_2)$. He/she is individual 1 (resp. individual 2), if $\min(X_1, X_2) = X_1$ [resp. $\min(X_1, X_2) = X_2$]. After this event, there can be a change in the potential residual lifetime distribution of the surviving individual. The potential residual lifetime of individual 1 (resp. individual 2) after the death of individual 2 (resp. individual 1) is denoted by X_3 (resp. X_4).

The joint distribution of the four latent variables is characterized by

(i) the joint survival function of (X_1, X_2):

$$S_{12}(x_1, x_2) = \mathbb{P}[X_1 > x_1, X_2 > x_2]; \tag{2.1}$$

(ii) the survival function of X_3 given $X_2 = \min(X_1, X_2) = z$:

$$S_3(x_3; z) = \mathbb{P}[X_3 > x_3 | X_2 = \min(X_1, X_2) = z]. \tag{2.2}$$

(iii) The survival function of X_4 given $X_1 = \min(X_1, X_2) = z$:

$$S_4(x_4; z) = \mathbb{P}[X_4 > x_4 | X_1 = \min(X_1, X_2) = z]. \tag{2.3}$$

These three joint and conditional survival functions, defined on $(0, \infty)$, characterize the latent model for the analysis of the mortality in the couple. In this model there exist at least three generation effects corresponding to the generations of each spouse, and to the generation of the couple, respectively.

2.2. Individual lifetimes

2.2.1. Link between the individual lifetimes and the latent variables

The lifetimes of individuals 1 and 2 (since the beginning of the common life) are denoted by Y_1 and Y_2. They can be expressed in terms of the latent variables as:

$$\begin{cases} Y_1 = X_1 \mathbb{1}_{X_1<X_2} + (X_2 + X_3)\mathbb{1}_{X_2<X_1} \\ \quad = \min(X_1, X_2) + X_3 \mathbb{1}_{X_2<X_1}, \\ Y_2 = X_2 \mathbb{1}_{X_2<X_1} + (X_1 + X_4)\mathbb{1}_{X_1<X_2} \\ \quad = \min(X_1, X_2) + X_4 \mathbb{1}_{X_1<X_2}. \end{cases} \tag{2.4}$$

This system can be partially solved. First, the X_1, X_2 variables are related to variables (Y_1, Y_2):

$\min(Y_1, Y_2) = \min(X_1, X_2)$, and

$Y_1 > Y_2$, if and only if $X_1 > X_2$.

Then the variables X_3 and X_4 can be deduced in some regimes[2] since:

$X_3 \mathbb{1}_{Y_2<Y_1} = Y_1 - \min(Y_1, Y_2)$ and

$X_4 \mathbb{1}_{Y_1<Y_2} = Y_2 - \min(Y_1, Y_2)$.

As noted in Norberg (1989), the observed model can be interpreted in terms of a chain with four possible states,[3] that are:

- state 1: both spouses are alive,
- state 2: husband dead, wife alive,
- state 3: husband alive, wife dead,
- state 4: both spouses are dead,

and transitions can only arise between states 1 and 2, 1 and 3, 2 and 4, and 3 and 4. Since the mortality intensity of a spouse can depend not only on the current state, but potentially on the time elapsed since the death of the other spouse, we get an example of a semi-Markov chain.

2.2.2. The joint density function and its decomposition

The joint probability density function (pdf) of (Y_1, Y_2) is easily derived from the distribution of the latent variables. We have (see Appendix A):

$$\begin{aligned} f(y_1, y_2) &= \left[-\frac{\partial S_{12}}{\partial x_1}(y_1, y_1)\right]\left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1)\right], \\ &\quad \text{if } y_2 > y_1, \\ &= \left[-\frac{\partial S_{12}}{\partial x_2}(y_2, y_2)\right]\left[-\frac{\partial S_3}{\partial x_3}(y_1 - y_2; y_2)\right], \\ &\quad \text{if } y_1 > y_2. \end{aligned} \tag{2.5}$$

Therefore, the joint density function can feature a discontinuity when $y_1 = y_2$.

Let us consider the case $y_2 > y_1$. The density can also be written as:

$$\begin{aligned} f(y_1, y_2) = -\frac{\partial S^*}{\partial y}(y_1)\left[\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) \Big/ \frac{\partial S^*}{\partial y}(y_1)\right] \\ \times \left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1)\right], \end{aligned} \tag{2.6}$$

where $S^*(y) = S_{12}(y, y)$ is the survival function of $\min(X_1, X_2)$ and $\frac{\partial S^*}{\partial y}(y) = \frac{\partial S_{12}}{\partial x_1}(y, y) + \frac{\partial S_{12}}{\partial x_2}(y, y)$. Thus, the decomposition of the bivariate density involves three components:

(i) $\left[-\frac{\partial S^*}{\partial y}(y_1)\right]$ is the density of the first death event;

(ii) the ratio $\left[\frac{\partial S_{12}}{\partial x_1}(y_1, y_1) / \frac{\partial S^*}{\partial y}(y_1)\right]$ is the probability that individual 1 dies at this first death event. It is equal to:

$$\mathbb{P}[Y_1 < Y_2 | \min(Y_1, Y_2) = y_1],$$

(iii) $\left[-\frac{\partial S_4}{\partial x_4}(y_2 - y_1; y_1)\right]$ is the density of the residual lifetime after this event.

2.2.3. Individual mortality intensities

Let us now derive the individual mortality intensities given the current information concerning the couple. Their expressions depend on the state either alive, or dead, of the other spouse.

(i) Let us first consider a date y at which both individuals are still alive, that is, such that $Y_1 \geq y$, $Y_2 \geq y$. The mortality intensity of individual 1 is defined by:

$$\begin{aligned} &\lambda_1(y | Y_1 \geq y, Y_2 \geq y) \\ &= \lim_{dy \to 0^+} \left\{\frac{1}{dy} P[y \leq Y_1 \leq y + dy | Y_1 \geq y, Y_2 \geq y]\right\} \\ &= \int_y^\infty f(y, y_2) dy_2 / S^*(y). \end{aligned} \tag{2.7}$$

[2] There are two regimes, corresponding respectively to the cases $Y_1 < Y_2$ and $Y_2 < Y_1$.

[3] In their analysis (Ji et al., 2011) consider also the possibility of a direct transition from state 1 to state 4 to account for catastrophic events (car accidents, plane crash) implying simultaneous deaths. They use a 5-day cut-off to account for a possible lag in reporting.