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An order of asymmetry in copulas, and implications for risk management

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ABSTRACT

We study symmetry properties of bivariate copulas. For this, we introduce an order of asymmetry, as well as measures of asymmetry which are monotone in that order. In an empirical study, we illustrate that asymmetric dependence structures do indeed occur in financial market data and discuss its relevance for financial risk management.

1. Introduction and results

Modeling the dependence structure between random variables is essential for finance and risk management. In practice, this leads to the problem of fitting a copula to a given set of data, which is mostly addressed by choosing an 'appropriate' parametric family of bivariate copulas and finding the 'correct' parameter. Obviously, the choice of the copula family determines the goodness of the fit and the predictions in a fundamental way. Nowadays, the practitioner has a variety of parametric copula families at hand, the most prominent of which are Archimedean copulas.

However, most of these families are unable to incorporate a fundamental feature of the data and, hence, are not optimal for applications. This fundamental feature is asymmetry, by which we mean the fact that $C(u, v)$ may not be the same as $C(v, u)$.[3] Most standard copula models, in particular all Archimedean copulas, are symmetric, i.e., they satisfy $C(u, v) = C(v, u)$ for all u, v. Therefore, if a given set of data possesses some degree of asymmetry, these standard models are not adequate, and asymmetric copula models should be taken into consideration. It should be noted, however, that there are also asymmetric models, e.g., the so-called Liouville copulas introduced in McNeil and Nešlehová (2010).

Asymmetry of copulas has been considered, for instance, in Klement and Mesiar (2006) and Nelsen (2007) where the copulas with the largest measure of asymmetry are identified. However, without an underlying order of asymmetry, results of this kind can be misleading; see Remark 2.19. Only recently symmetry and asymmetry of copulas have been considered from a statistical point of view: Genest et al. (2012) constructed a test of global symmetry and applied it to nutrition data, whereas Kojadinovic and Yan (2012) only considered the special case of extreme value copulas and applied their test to insurance data. Perhaps most surprisingly, several studies in the finance and econometrics literature (see, e.g., Christoffersen et al., 2012, 2013) have found the skewed t-copula of Demarta and McNeil (2004) to provide a good fit to financial market data with the skewness parameter that governs the copula's degree of asymmetry regularly being significantly

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3 Note that the term asymmetry is also used in a different context, namely when referring to the different behavior of the lower and upper tail dependence.

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different from zero. However, none of these studies discusses this apparently unexplored stylized fact of financial data.

The aim of this paper is twofold. First, we want to develop a consistent theory of asymmetry in bivariate distributions. To be able to quantify the degree of asymmetry of a copula, we introduce an order of asymmetry for copulas and define several measures which are monotone in that order. It is important to understand that measures without an underlying order are not sufficient for comparing the degree of asymmetry of two given copulas. Indeed, one might be tempted to call a copula C less asymmetric than another copula D provided $\mu(C) \le \mu(D)$ for some reasonable asymmetry measure μ. However, there will often exist other, equally reasonable measures of asymmetry satisfying the reverse inequality. If, on the other hand, we can define an order $\prec$ of asymmetry we know a copula C is less asymmetric than D if, and only if, $C \prec D$. In this case, all measures μ that are monotone in the order $\prec$ (i.e., $\mu(C) \le \mu(D)$ whenever $C \prec D$) are consistent with the concept of asymmetry and do not lead to misinterpretations. Of course, all said applies equally well to any other situation where one wants to quantify a certain feature—the use of measures without an underlying order is prone to creating mistakes.

The second goal of our paper is to show that asymmetry does indeed occur empirically in financial data. We start by giving an intuitive interpretation of asymmetric dependence as better diversification against downturns of a reference asset (e.g., the market portfolio). We then employ the test of Genest et al. (2012) which is based on a measure of asymmetry that is monotone in our order and test a sample of financial return series for the presence of asymmetry in the all possible bivariate return series pairs. The results that we find show that pairs of assets built across different asset classes are indeed characterized by significant asymmetry in their dependence.

The rest of the paper is organized as follows. In the central Section 2, we introduce an order of asymmetry, investigate its properties, and construct several measures of asymmetry monotone in that order. Then, Section 3 illustrates that asymmetry does indeed occur in different sets of financial data.

2. An order and nonparametric measures of asymmetry

2.1. An order of asymmetry for copulas

Consider two real-valued random variables X and Y on some probability space. Then X and Y are called exchangeable if (X, Y) and (Y, X) have the same distribution, i.e. if their joint distribution function $F_{X,Y}$ is symmetric:

$$F_{X,Y}(x, y) = F_{X,Y}(y, x).$$

Note that exchangeable random variables are necessarily identically distributed.

We assume that the univariate margins F_X and F_Y are continuous, and denote the copula of (X, Y) by $C(X, Y)$; see Nelsen (2006) for an introduction to and more details about copulas. Then, exchangeability of two identically distributed random variables X and Y is equivalent to the symmetry of their copula $C_{X,Y}$:

$$C_{X,Y} = C^{\top}_{X,Y},$$

where the transpose of a copula C is defined as $C^{\top}(u, v) := C(v, u)$.

Definition 2.1. A copula C is called symmetric if $C = C^{\top}$, and asymmetric otherwise.

Definition 2.2. A copula C is said to be less asymmetric than a copula D, written $C \prec D$, if and only if

$$|C(u, v) - C(v, u)| \le |D(u, v) - D(v, u)|$$

for all $(u, v) \in I^2$. We call $\prec$ the *order of asymmetry* on the set of copulas.

Remark 2.3. 1. Note that $C \prec D$ is the same as saying that

$$|C - C^{\top}| \le |D - D^{\top}|$$

pointwise in I^2.

2. It is easy to see that the relation $\prec$ is reflexive and transitive. However, it is not antisymmetric since $C \prec D$ and $D \prec C$ is equivalent to $|C - C^{\top}| = |D - D^{\top}|$ which does not imply that $C = D$ (consider, for instance, symmetric C and D). Therefore, $\prec$ is a preorder and not an order; nevertheless we will use the term 'order of asymmetry' for $\prec$.
3. $\prec$ is not total, i.e., there are copulas which cannot be ordered w.r.t. $\prec$.

Definition 2.4. Consider a set S with a preorder $\le$.

An element $m \in S$ is called a *maximal element* if $x \le m$ for all $x \in S$ which are comparable to m (i.e., which satisfy $x \le m$ or $m \le x$). An element $m \in S$ is called a *greatest element* if $x \le m$ for all $x \in S$.

Analogously, one defines minimal, respectively smallest, elements by replacing $\le$ by $\ge$.

Remark 2.5. It is clear that any greatest element is also maximal; the converse, in general, false.

Note that, since we are dealing with preorders instead of orders, there may be more than one greatest element. Note that $C \prec D \prec C$ just means that $|C - C^{\top}| = |D - D^{\top}|$ everywhere, which does not mean $C = D$ (consider, for instance, $D = C^{\top}$ for an asymmetric C).

The following results state, loosely speaking, that the symmetric copulas are precisely the least asymmetric copulas, but that there is no maximally asymmetric copula.

Proposition 2.6. *Each symmetric copula is a smallest (hence a minimal) element w.r.t. $\prec$, and there are no other minimal elements.*

Proof. The first assertion follows from the fact that each symmetric copula C is comparable to any other copula D, and satisfies $C \prec D$. The fact that any minimal element must be comparable to all symmetric copulas implies that any minimal element must be symmetric itself. ∎

Theorem 2.7. *There is no greatest element w.r.t. $\prec$.*

Proof. Klement and Mesiar (2006) have shown that for any copula C we have

$$0 \le |C(u, v) - C(v, u)| \le \min(u, v, 1 - u, 1 - v, |u - v|) \qquad (1)$$

for every $(u, v) \in I^2$, and that for each $(u, v) \in I^2$ there exists a copula C such that $|C(u, v) - C(v, u)| = \min(u, v, 1 - u, 1 - v, |u - v|)$. Hence, if D is a greatest element w.r.t. $\prec$ we have $|D(u, v) - D(v, u)| = \min(u, v, 1 - u, 1 - v, |u - v|)$ for every $(u, v) \in I^2$. But it is also proven in Klement and Mesiar (2006) that there does not exist a copula D with the above property. ∎

The next result states that $\prec$ is invariant under transposition and survival operation. Here, given a copula C, its *survival copula* $\hat{C}$ is defined by

$$\hat{C}(u, v) := u + v - 1 + C(1 - u, 1 - v).$$

If C corresponds to the distribution function $F_{X,Y}$, its survival copula corresponds to the joint survival function $\hat{F}(x, y)$ given by $\hat{F}(x, y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y)$.