Insurance: Mathematics and Economics 69 (2016) 138–148

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Applications of central limit theorems for equity-linked insurance

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ARTICLE INFO

Article history:
Received December 2015
Received in revised form
April 2016
Accepted 12 May 2016
Available online 19 May 2016

Keywords:
Equity-linked insurance
Variable annuity guaranteed benefits
Risk measures
Strong law of large numbers
Central limit theorem
Individual model
Aggregate model

ABSTRACT

In both the past literature and industrial practice, it was often implicitly used without any justification that the classical strong law of large numbers applies to the modeling of equity-linked insurance. However, as all policyholders' benefits are linked to common equity indices or funds, the classical assumption of independent claims is clearly inappropriate for equity-linked insurance. In other words, the strong law of large numbers fails to apply in the classical sense. In this paper, we investigate this fundamental question regarding the validity of strong laws of large numbers for equity-linked insurance. As a result, extensions of classical laws of large numbers and central limit theorem are presented, which are shown to apply to a great variety of equity-linked insurance products.

1. Introduction

One of the most fundamental principles for the insurance business is the pooling of funds from a large number of policyholders to pay for losses that a few policyholders incur. The mathematics behind such a business model is the law of large numbers which dictates that actual average loss would be close to the theoretical mean of loss with a large pool of homogeneous and independent risks. Take for example a traditional pure endowment life insurance that pays a lump sum of B dollars upon survival at the end of T years. Suppose that an insurer sells identical policies to n policyholders all of who are of the same age x. We denote the future lifetime of the ith policyholder by $\tau_x^{(i)}$ and the survival probability by ${}_Tp_x := \mathbb{P}(\tau_x^{(i)} > T)$. Even though there is uncertainty to each contract with regard to whether the policyholder survives at time T, the strong law of large numbers implies that the percentage of survivorship is almost certain and so is the average benefit payment for each contract, i.e.

$$\frac{1}{n}\sum_{i=1}^{n} BI(\tau_x^{(i)} \geq T) \longrightarrow \mathbb{E}[BI(\tau_x^{(1)} \geq T)] = B_Tp_x, \quad n \to \infty.$$

Simply put, this is the scientific ground of the insurance practice of pricing and reserving, which mandate a fixed amount of liquid asset on the aggregate level to pay for seemingly uncertain benefits on individual basis. In other words, the mortality risks involved in all individual contracts are diversified through the pooling of a large number of homogeneous contracts.

The past few decades have seen the rapid growth of investment-combined insurance products, which allow policyholders to reap the benefits of equity investment on their premiums. Insurers around the world have developed a variety of equity-linked insurance. However, this market innovation brought financial risks into insurance contracts, in conjunction with traditional mortality risk. Since policy benefits are often linked to the same equity-indices or funds, there is no diversification of financial risks amongst each cohort of policyholders. If the equity market performs poorly, there is an erosion of policy values to all contracts at the same time. From the viewpoint of statistics, the individual policy benefits are no longer independent random variables and hence classical laws of large numbers do not apply.

Nevertheless, it is fairly common both in practice and in the actuarial literature that mortality risks are implicitly assumed to be diversified in premium and reserve calculations for equity-linked insurance. See examples in Sections 3.2 and 4.1–4.3. Is there any theoretical basis of such a widely used assumption? In this paper, we intend to address this question and explore various sets of assumptions under which the law of large numbers can be extended to equity-linked insurance. In Section 2, we provide

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http://dx.doi.org/10.1016/j.insmatheco.2016.05.004
0167-6687/

a concrete example to quantify and analyze the interaction of mortality and financial risks. We extend laws of large numbers to a general framework of survival benefits in Section 3 and to that of death benefits in Section 4. We conclude the paper with a numerical example for the central limit theorem result and its application to the calculation of risk measures.

2. Guaranteed minimum maturity benefit

In this section, we take the guaranteed minimum maturity benefit (GMMB) as a model example, although similar results can also be obtained for all other types of equity-linked insurance to be introduced in later sections. In Feng and Volkmer (2012), the net liability of the GMMB is defined to be the present value of future outgo less the present value of future income on a standalone contract basis. We shall describe the cost and benefit of the GMMB rider for the sake of completeness. Let $L(\tau^{(i)}),\ i \in \{1, 2, \ldots, n\}$ be the GMMB net liability for the ith policyholder. Assume all policyholders make the same amount of initial purchase payment at the policy issue (this assumption will be loosen in the next section). Denote the guaranteed maturity benefit by G and the evolution of investment accounts by $\{F_t : t \geq 0\}$, which is modeled by some stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t : t \geq 0\})$. Throughout the paper, we assume $\{F_t : t \geq 0\}$ to be a non-negative process. Rider fees and charges are collected continuously as a fixed percentage m_e of the account values. Let r be the annual yield rate on bonds backing up the liabilities. The GMMB rider is an equity-linked insurance analog of the traditional pure endowment insurance. It offers a policyholder the greater of a minimum value and account value, should the policyholder survive the maturity T. Since the bulk of the guaranteed amount G comes out of the policyholder's own investment account, the out-of-pocket cost of the GMMB rider for the insurer is determined by

$$e^{-rT}(G - F_T)_+ I(\tau_x^{(i)} > T),$$

where $(x)_+ = \max\{x, 0\}$. To compensate for the GMMB liability, the insurer receives a continuous flow of fee incomes until the earlier of policyholder's death and the maturity, the present value of which is given by

$$\int_0^{T \wedge \tau_x^{(i)}} e^{-rs} m_e F_s \, ds.$$

Then the present value of *individual net liability* (gross liability less fee income) is given by

$$L(\tau^{(i)}) := e^{-rT}(G - F_T)_+ I(\tau_x^{(i)} > T) - \int_0^{T \wedge \tau_x^{(i)}} e^{-rs} m_e F_s \, ds. \tag{1}$$

It was also shown in Feng (2014) that the actual model used by practitioners through spreadsheet calculations is the *average net liability* model

$$L^* := \mathbb{E}\big[L(\tau^{(i)}) \big| \mathcal{F}_T\big] = {}_T p_x e^{-rT}(G - F_T)_+ - \int_0^T {}_s p_x e^{-rs} m_e F_s \, ds. \tag{2}$$

Observe that there are two sources of randomness, namely $\tau_x^{(i)}$ and $\{F_t : t \geq 0\}$ in the formulation of net liability in (1), whereas only financial risk is present in the formulation of net liability in (2). Before discussing the connection between these two types of models, we digress to investigate the effect on the tail behavior of undiversifiable risks.

Let us consider a set of n pairwise symmetric random variables, i.e. (X_i, X_j) has the same distribution for all $i, j = 1, \ldots, n$. Note that X_i and X_j do not need to be independent. The pairwise symmetry is equivalent to the statement that $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ has the same distribution as $(X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n)$ for any $i, j = 1, \ldots, n$.

Proposition 2.1. *Let $\{X_1, X_2, \ldots\}$ be a set of pairwise symmetric random variables. For any positive integer n,*

$$\frac{1}{n+1}\sum_{i=1}^{n+1} X_i \leq_{cx} \frac{1}{n}\sum_{i=1}^{n} X_i, \tag{3}$$

where $X \leq_{cx} Y$ means that $\mathbb{E}g(X) \leq \mathbb{E}g(Y)$ for any convex function g.

Proof. For any convex function g, observe that

$$\mathbb{E}\left[g\left(\frac{1}{n+1}\sum_{i=1}^{n+1} X_i\right)\right] = \mathbb{E}\left[g\left(\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{n}\sum_{j=1, j\neq i}^{n+1} X_j\right)\right].$$

Applying Jensen's inequality to a discrete random variable, we can show that for any convex function g and $x_1, x_2, \ldots, x_n \in \mathbb{R}$,

$$\frac{1}{n}\sum_{i=1}^{n} g(x_i) \geq g\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[g\left(\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{n}\sum_{j=1, j\neq i}^{n+1} X_j\right)\right] \\
&\quad \leq \frac{1}{n+1}\sum_{i=1}^{n+1}\mathbb{E}\left[g\left(\frac{1}{n}\sum_{j=1, j\neq i}^{n+1} X_j\right)\right] \\
&\quad = \frac{1}{n+1}\sum_{i=1}^{n+1}\mathbb{E}\left[g\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)\right],
\end{aligned}$$

where the last equality follows from the fact that $(X_1, \ldots, X_n)$ are pairwise symmetric. Therefore, we have proved that for any convex function g

$$\mathbb{E}\left[g\left(\frac{1}{n+1}\sum_{i=1}^{n+1} X_i\right)\right] \leq \mathbb{E}\left[g\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)\right],$$

which establishes the convex order in (3). □

It follows immediately that for $p \in (0, 1)$ (cf. Dhaene et al., 2006, Theorem 3.2),

$$\mathrm{TVaR}_p\left(\frac{1}{n+1}\sum_{i=1}^{n+1} X_i\right) \leq \mathrm{TVaR}_p\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right).$$

The tail-value-at-risk of the average of n losses is a decreasing function of the sample size n. In other words, the tail risk of average loss can always be reduced by diversification through a large pool of policies. When the components of X are independent, then the strong law of large numbers implies that the limit of $\mathrm{TVaR}_p[(1/n)\sum_{j=1}^{n} X_j]$ is $\mathbb{E}[X_j]$ as $n \to \infty$. However, this is not true in general when the components are dependent, as in the case of equity-linked insurance contracts. A discussion of systematic versus diversifiable risks can be found in Busse et al. (2014) with detailed numerical examples. The above provides a theoretical justification of their observations. Denuit et al. (2005, Proposition 3.4.23) provides a special case of Proposition 2.1 which requires the assumption of independent and identical distributed risks.

As alluded to earlier, individual net liabilities $(L(\tau^{(1)}), L(\tau^{(2)}), \ldots, L(\tau^{(n)}))$ are not mutually independent. Hence it is critical for